INVESTIGATION OF STICK BOMBING TACTICS AGAINST RECTILINEAR TARGETS

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THESIS

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ABSTRACT

A mathematical model for the delivery of conventional bombs by stick bombing has been developed. This model considers bomb type, number of bombs, pilot miss-distance distribution, and dimensions of a rectilinear target. It may be used to estimate the effects of bomb interval and approach angle upon the single pass destruction probability of a target.

The significant variables of the stick bombing operation are identified, and functional relationships are developed among these. The model is used to examine the problem of determining an optimal approach angle and bomb interval for stick bombing bridges and railroads.

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I. STATEMENT OF PROBLEM

During the Vietnam conflict the U. S. Navy was given the primary responsibility of slowing the flow of supplies from North Vietnam to South Vietnam. This task was partially accomplished by cutting railroads, roads, and bridges on all North-South avenues of travel in North Vietnam. The major weapons systems used were light jet aircraft* carrying conventional general purpose bombs.*

Since this type of combat bombing had last been utilized during the Korean War and in fact had only been practiced as a secondary mission (the primary mission of the squadrons involved had been nuclear strike) limited knowledge and know how existed on how best to attack these targets. Furthermore the available publications at that time failed to provide adequate answers. In many cases the publications discussed targets of vastly different dimensions and construction than those encountered in North Vietnam and the recommendations advanced in these publications failed to fully incorporate the advantages of stick bombing.*

The pilots, therefore, had to develop their own estimate of the best tactics to use. These tactics were largely based on "seat of the pants" estimates and had little if any analytical evidence to support them. It is the purpose of this thesis to provide the necessary analytical techniques.

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*All terms with asterisks are defined in Appendix E

In order to discuss the tactics developed, the variables available to the pilot must first be delineated. Once the target has been selected the pilot has some control over the following variables: (1) the number of airplanes used to attack the target, (2) the actual order and interval between airplanes in the attack, (3) the number of passes per airplane over the target, (4) the number and type of weapons that the airplanes carry, (5) the approach angle* to the target, (6) the interval* between weapons and hence the stick length*, (7) the maximum and minimum altitudes of releasing the weapons, (8) the dive angle* and speed in the dive.[1]

Although tactics varied somewhat from airwing* to airwing and even squadron* to squadron, an accurate description of the generally accepted typical tactic used to cut either a road, railroad, or bridge is as follows: for a given target 3 or 4 airplanes loaded with 4 to 6 500 lb. general purpose bombs* were sent as a flight*. The aircraft proceeded in formation to the target vicinity whereupon the flight leader identified the target and rolled in*, followed in approximately 5 seconds and with a different approach angle by the second airplane and then in a similar manner by the third airplane, etc. Each pilot made an individual tracking* dive attempting to dive at a predetermined dive angle and speed. At a preselected altitude each pilot released all of his bombs in a stick of fixed and equal bomb intervals and then departed the target area.

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[1]Joint Munitions Effectiveness Manual (Air to Surface) Weapons Effectiveness, Selection, and Requirements, 1A-4, Unclassified, 5 April 1968.

The questions immediately raised when examining these tactics are: for a given bomb load, what are the effects of changing the approach angle on the probability of a single pass hit* (Psph)? What does the interval between bombs depend on and how can it be set so as to maximize the Psph? What effects do target dimensions have on the tactics developed? And lastly since pilot skill varies from pilot to pilot, how does changing the pilot miss-distance distribution* affect the overall results?

In attempting to answer these questions, several of the pilot controlled variables are assumed fixed throughout the investigation. In the first place it is assumed that the airplanes attack separately and make individual tracking runs; the dive angle, release speed, maximum and minimum release altitudes are invariant for all tactics investigated; and lastly all weapons are dropped on a single pass.

II. DEVELOPMENT OF THE GENERAL MODEL

To answer the above questions relating to tactics, an operational model of the weapon system will be developed in the following sequential manner: analyze the operation and then translate these results into a mathematical model.

In analyzing the operation of the weapon system (light jet attack aircraft carrying conventional general purpose bombs) it seems appropriate to address the following topics:

a. What are the significant parameters of weapon system performance in its operational environment?

b. How is system performance related to these parameters (what functional relationships)?

c. Parameters used in any mathematical model subsequently developed should be capable of being estimated with empirical data.

The above operations analysis will be used to generate a mathematical model. This model will be used to gain insight into the performance of the system and determine how to optimize system effectiveness according to the criterion of single pass destruction probability.

Thus, we must start with an analysis of the bombing operation. The entire sequence of operations from target detection to munition detonation was considered. System performance was partitioned into the following subsystems:

a. Acquisition

b. Delivery

c. Effects

Acquisition was not considered further since it is a factor held constant in this investigation. System delivery capability was further subdivided as follows:

a. from detection to bomb release,

b. from bomb release to detonation.

The first aspect is discussed in the section, "Analysis of Aircraft Attack from Roll-in to Weapon Release", while the second is discussed in the section, "Analysis of Pilot Miss-distance Distribution". These various aspects of the bombing attack will now be discussed. It should be noted that the analysis was based not only upon the author's 3-1/2 years in a light jet attack squadron during which time many missions were flown against targets of this type but also upon discussions with many pilots of similar backgrounds.

A. ANALYSIS OF PILOT MISS-DISTANCE DISTRIBUTION

In the first place, miss-distance was defined as the error between the actual point of impact of the bomb center and the intended point of impact of the bomb center. An orthogonal coordinate system was defined along and perpendicular to the flight path of the aircraft with its origin at the intended point of impact of the bomb center or stick center if more than one weapon were dropped. The error was then decomposed into two components: range error (R) along the flight path and deflection error (D) perpendicular to the flight path. It was assumed that range error and deflection error were statistically

independent with a joint density function of the bi-variate normal form,[2]

$$f_{\underline{R}\,\underline{D}}(R,D) = \frac{1}{2\pi\sigma_R\sigma_D} \exp\left\{-\frac{1}{2}\left[\left(\frac{R-\mu_R}{\sigma_R}\right)^2 + \left(\frac{D-\mu_D}{\sigma_D}\right)^2\right]\right\}.$$

For a random variable such as range error which is composed of many factors (wind, target acquisition, bomb release, etc.), a postulated normal distribution is reasonable in light of the Central Limit Theorem. This assumption was also consistent with available empirical data.

Since all pilots keep extensive records of bomb hits made during practice bomb runs on instrumented targets, pilots were assumed to know their expected range error and deflection error and to be able to correct for it, that is $\mu_R = \mu_D = 0$. This assumption is obviously optimistic but it guarantees that all results obtained for Psph are upper bounds and it certainly is the ideal toward which pilots work.

Defined in this manner, the pilot miss-distance distribution can be characterized by two parameters, namely σ_R and σ_D, and by varying σ_R and σ_D, the effect of the pilot miss-distance distribution on Psph can be observed.

B. ANALYSIS OF AIRCRAFT ATTACK FROM ROLL-IN TO WEAPON RELEASE

The parameters that affect the attack from roll-in to weapon release are:

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[2] Anderson, T. W., An Introduction to Multivariate Statistical Analysis, p. 11, Wiley, 1958.

Dive Angle

Release Speed

Maximum Release Altitude

Minimum Release Altitude

Approach Angle

Bomb Interval

In order to examine different tactics available, a standard bomb run was defined and used throughout the investigation. That is to say dive angle, release speed, maximum and minimum release altitudes were all held constant. The purpose of defining this standard bomb run was to maximize Psph for this standard. Furthermore, these parameters actually control the pilot miss-distance distribution. Hence, if the examination of different types of bomb runs is desired, only the relationship between these parameters and the pilot miss-distance distribution need be determined before the analysis is performed.

Approach angle was defined as the acute angle between the center line of the target and the horizontal projection of the flight path. During the investigation the approach angle was allowed to vary from $0 - 90^{\circ}$. It was assumed that the approach angle was selected prior to the bombing run and that the pilot attempted to make his attack using this approach angle. In order to account for pilot error in this parameter, a known probability density function, $g_{\theta}(\theta)$, of approach angle error (AAE) was assumed.

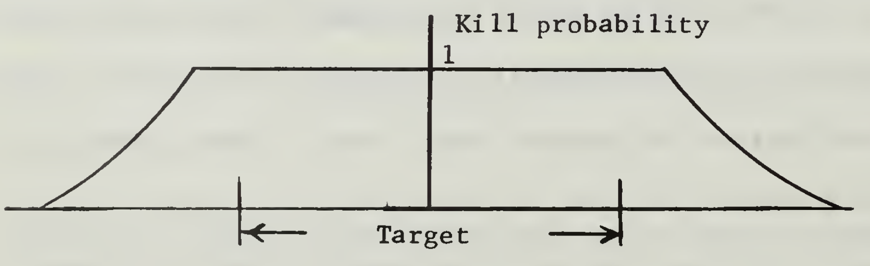
Lastly the only constraint placed on the selection of bomb interval (INT) was that for any bomb stick the intervals must be equal.

With these assumptions, it is possible to mathematically describe the attack from roll-in to weapon release by three parameters: the number of bombs to be dropped (n), the preselected approach angle (θ_0), and the bomb interval (INT).

C. ANALYSIS OF MUNITION EFFECTIVENESS

The weapons and the weapon release mechanisms were assigned a reliability factor of 1. For a comparative evaluation of tactics this is reasonable, however, an analysis of the effect of relaxing this assumption was made and a discussion of it can be found in Chapter IV.

A convenient method of describing weapons effects is to define a lethality function, L(d), where the lethality function is a description of the probable damage caused as a function of miss-distance. For example, for a one dimensional target the lethality function might take the form of Figure 1.

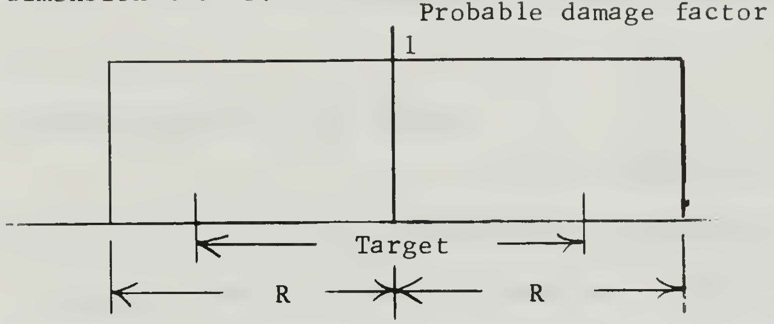


Typical one dimensional lethality function

FIGURE 1

For this model a "cookie cutter" type lethality function was assumed. Although it has been noted the lethality of a fragmenting

munition obeys an exponential decay with radial miss-distance[3], the exploratory nature of this investigation seemed to justify the simpler model. Thus the probability of killing* the target is 1 if the miss-distance is less than or equal to R and 0 otherwise. See Figure 2 for the one dimensional case.



One dimensional "cookie cutter" lethality function

FIGURE 2

By using this type of lethality function the weapon effectiveness can be characterized by one parameter. In this case the parameter is defined as the effective miss-distance (EMD) and is the maximum distance from any target dimension that the center of impact of a weapon may be and still produce a kill.

D. MATHEMATICAL FORMULATION

The problem has now been reduced to investigation of the effect on Psph of 6 parameters, namely

n = number of bombs dropped,

σ_R = standard deviation in range error,

σ_D = standard deviation in deflection error,

INT = bomb interval,

---

[3] Ballistics Research Lab Report 697, Justification of an Exponential Fall Off Law for Numbers of Effective Fragments, by H. W. Weiss, February, 1949.

θ_0 = preselected approach angle,

EMD = effective miss-distance,

and one arbitrarily defined probability density function, the approach angle error density, $g_{\underline{\theta}}(\theta)$.

In the first place, the number of weapons (n) was assumed known and given for any attack. Hence for a given n

$$\max Psph = \iint\limits_{\text{all space}} \int\limits_{\text{all } \theta} L(R,D)\, f_{\underline{R}\,\underline{D}}(R,D)\, g_{\underline{\theta}}(\theta)\, dRdDd\theta,$$

where L(R,D) is the lethality function set equal to 1 when $(R,D)\epsilon A^*(\theta)$ (0 elsewhere) and $A^*(\theta)$ is the area, in the R,D plane, depending upon the approach angle, θ, over which the stick center may range and still achieve a target kill. Obviously $A^*(\theta)$ is also a function of EMD, INT, and target dimensions. In order to describe $A^*(\theta)$, however, the optimum bomb interval must first be determined.

As seen in Appendix A, by examining the geometry of the situation and the characteristics of the normal density function, the optimal bomb interval can be developed and is a function of EMD, θ and target dimensions. It is shown in Appendix A that the optimal bomb interval, INT_{opt}, is given by:

$$INT_{opt} = [2\ EMD + W]/\sin\theta.$$

In order to perform the integration, it is now necessary to determine the boundaries of A* in terms of the optimal bomb interval or in other words in terms of EMD, target dimensions and θ_0. This can be done but due to the complexity of the development and the differences in target shape, the details appear in Appendices B and C. Once the boundaries have been determined, however, and represented

in the R-D coordinate system

$$\text{MAX Psph} = \iint_{R,D \epsilon A^*} \int_{\text{All } \theta} g_{\underline{\theta}}(\theta)\, f_{\underline{R}\,\underline{D}}(R,D)\, dRdDd\theta$$

may be numerically integrated by the method discussed in Appendix D. Thus, optimal tactics may be determined by varying the parameters on which Psph depends.

III. APPLICATION OF MODEL IN THE DETERMINATION OF OPTIMAL TACTICS

A. RAILROAD/ROAD TARGET

1. Form of the Model

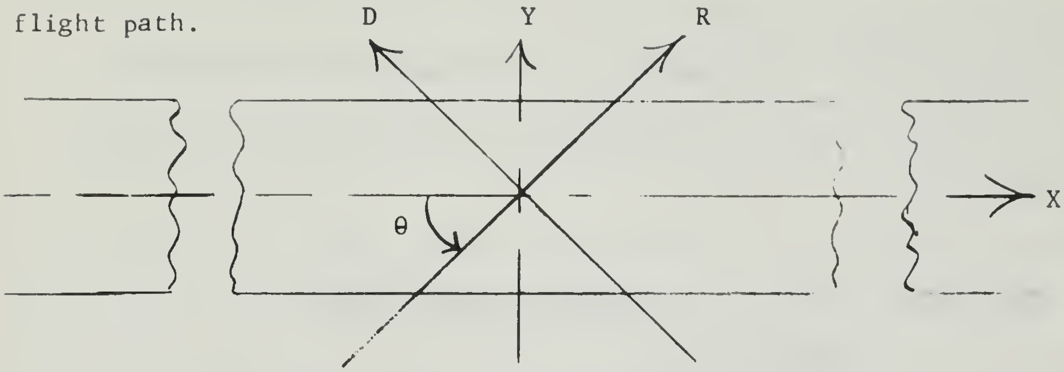
In this case the length of the target is assumed to be so great that it is effectively impossible to miss the target along its length dimension. This is equivalent to assuming that the railroad/road is an infinitely long rectilinear target. A straight segment of roadway approximately 1800' long is an example of a target which may be considered to be infinitely long. In order to show this suppose that the approach angle (θ) is 0^{o} and that only one weapon is to be dropped. Since the range error is always assumed to be larger than deflection error, this approach angle causes the length of the target to be most critical. Because 99.64% of the probability of a normal distribution lies between $\pm$ 3σ, $6\sigma_R$ will include essentially all the possible errors in the range direction. The largest value of σ_R used in this investigation was 300 feet; hence any target 1800 feet or longer will produce results that do not vary significantly from results obtained when a target of infinite length is used.

Suppose that the target has width (RRW), length ($L \longrightarrow \infty$), that the distribution is bi-variate normal with known σ_R and σ_D and $\mu_R = \mu_D = 0$,

$$f_{\underline{R}\ \underline{D}}(R,D) = \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^{\ 2}} + \frac{D^2}{\sigma_D^{\ 2}}\right)\right];$$

Define two coordinate systems, a X-Y coordinate axis parallel and perpendicular to the target dimensions and a R-D coordinate system

(used to measure pilot error) parallel and perpendicular to the flight path.



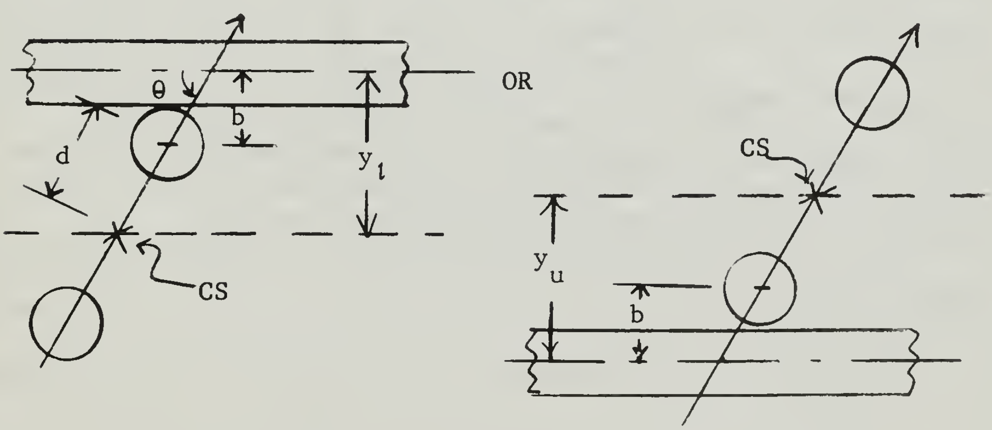
Coordinate axes in railroad/road problem

FIGURE 3

In order to reduce the notation assume that n = 2 (i.e., the stick consists of two weapons). Furthermore, for any given approach angle (θ), set the bomb interval (INT) for these weapons at the optimal value

$$INT = (2\ EMD + RRW)/\sin\theta.$$

Now, there is a target kill if the CS falls within certain limits. See Figure 4.



Limits of position of stick center needed
produce a target kill

FIGURE 4

where y_u = upper limit for CS and

y_l = lower limit for CS.

These limits of integration can be computed and expressed in the R-D coordinate system. See Appendix B, hence A* is well defined and

$$\text{MAX Psph} = \iint_{R,D \epsilon A^*} \int_{\text{All } \theta} f_{\underline{R}\ \underline{D}}(R,D)\ g_{\underline{\theta}}(\theta)\ dRdDd\theta$$

can be numerically evaluated for each θ_0, n, EMD, target size, and σ_R, σ_D.

2. Sample Results

The results presented in this section depict only several of the many possible combinations of parameters available. The results are presented in order to illustrate the possible functional dependence of angle and bomb interval. For this computer run the following values of the parameters were used: (1) bomb load = 6 (2) EMD = 12.5' (3) railroad width = 5' (4) $100' \leq \sigma_R \leq 300'$ (5) $50' \leq \sigma_D \leq 150'$.

Since no data was available to estimate the probability density function of approach angle error, the author assumed a discrete density function that, based on his experience, seemed representative. It was defined as follows for every preselected approach angle θ_0

$$\begin{aligned} g_{\underline{\theta}}(\theta) &= .1 \quad \theta_0 - 9^o \\ &= .1 \quad \theta_0 - 6^o \\ &= .2 \quad \theta_0 - 3^o \\ &= .2 \quad \theta_0 \\ &= .2 \quad \theta_0 + 3^o \\ &= .1 \quad \theta_0 + 6^o \end{aligned}$$

$$= .1 \quad \theta_0 + 9^o$$

where $\theta_0 \equiv$ pre-planned approach angle.

The optimum approach angle was found to be 8.62^o but Psph for this angle did not vary appreciably from the value obtained when $\theta = 10^o$. Notice that at 10^o, the rate of change of the bomb interval is relatively large; hence, the assumption about proper functioning of the bomb release equipment might be critical. On the other hand, when this particular distribution for Approach Angle Error (AAE) was assumed, the results obtained at $\theta_0 = 10^o$ are not too different from those when no error was assumed. The same type of results could have been obtained by assuming that there were no approach angle errors but that stick length was governed by an error distribution since both types of errors cause essentially the same results (i.e. a bomb interval is being used that is not optimal for the particular approach angle). Furthermore, notice that by assuming that stick length and approach angle are free from error although not necessarily causing erroneous results tend to inflate the values of Psph and to cause the maximum Psph to occur at too shallow an angle.

It is most important to remember that the values of Psph were all determined by calculating and using the optimum bomb interval for the approach angle or the E (Approach Angle) in the case of AAE and these values would be considerably different if for example, the bomb interval for $\theta = 80^o$ were used in calculating the Psph for $\theta = 10^o$.

Notice that Psph is almost constant for any given σ_D throughout the investigated range of σ_R. It is only when $\sigma_D = 50'$ that appreciable fall off occurs as σ_R increases. This indicates that if the standard deviation of pilot error in deflection is greater than

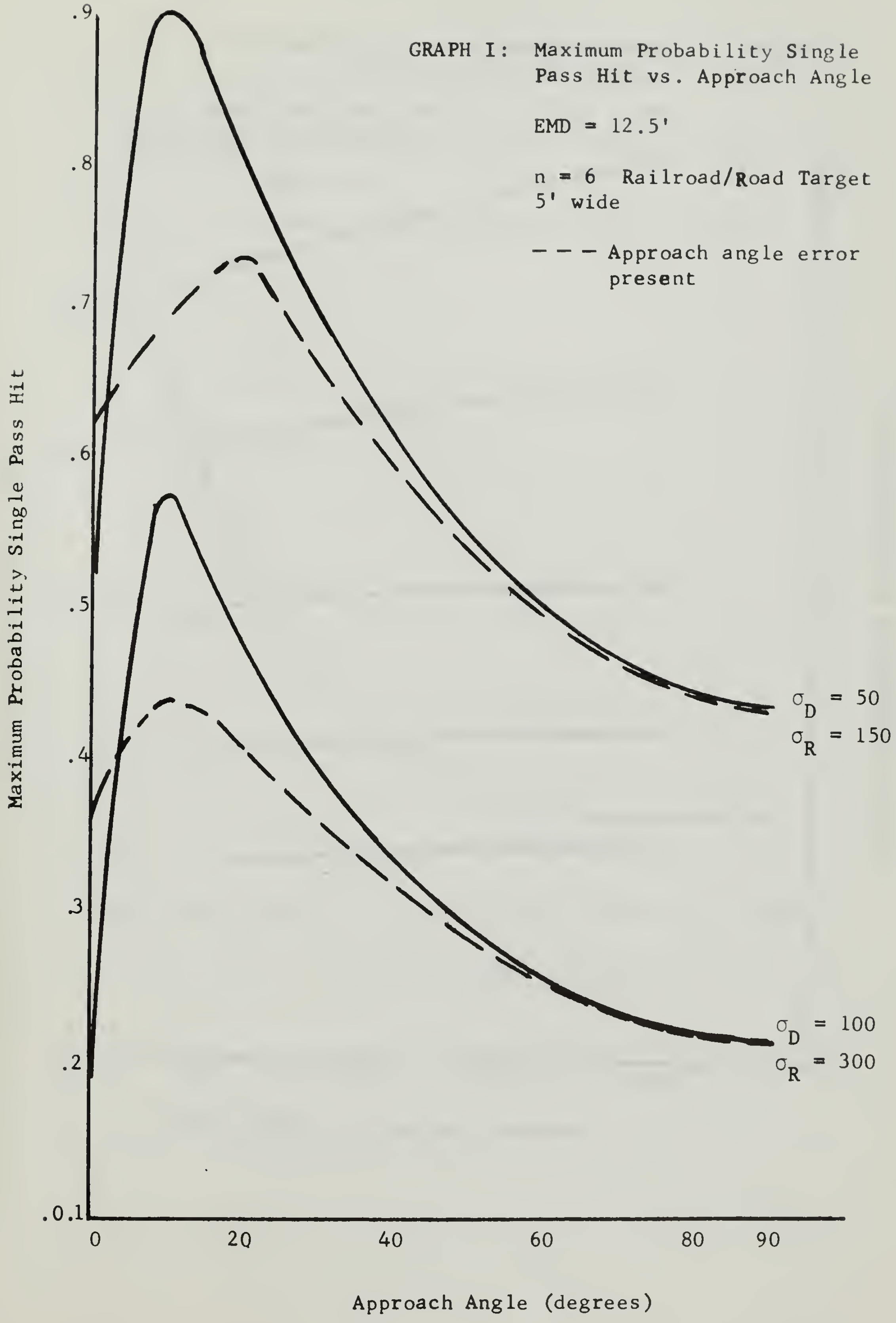
50', effort should be expended to reduce it prior to attempting to reduce σ_R. For example, suppose σ_D = 100' and σ_R = 300', then if σ_D is reduced by 50%, σ_R held constant, Psph increases from .60 to .83, but if σ_R is reduced by 50%, σ_D held constant, Psph only increases from .60 to .62.

Also it should be noted that approaching the target exactly along the center line of the target produces the lowest Psph. This of course is caused by the fact that at $\theta = 0^o$ all advantages of stick bombing are lost. Finally when θ is larger than 45^o, two interesting facts appear: (1) the bomb interval is almost constant and (2) when EAA is assumed to be present, the results compare almost exactly to those obtained when EAA is assumed absent. These results can be interpreted to mean that if errors in approach angle/bomb interval are thought to be large and unpredictable (i.e., the error distribution cannot be estimated) perhaps more consistent results can be obtained by using approach angles greater than 45^o.

B. BRIDGE TARGETS

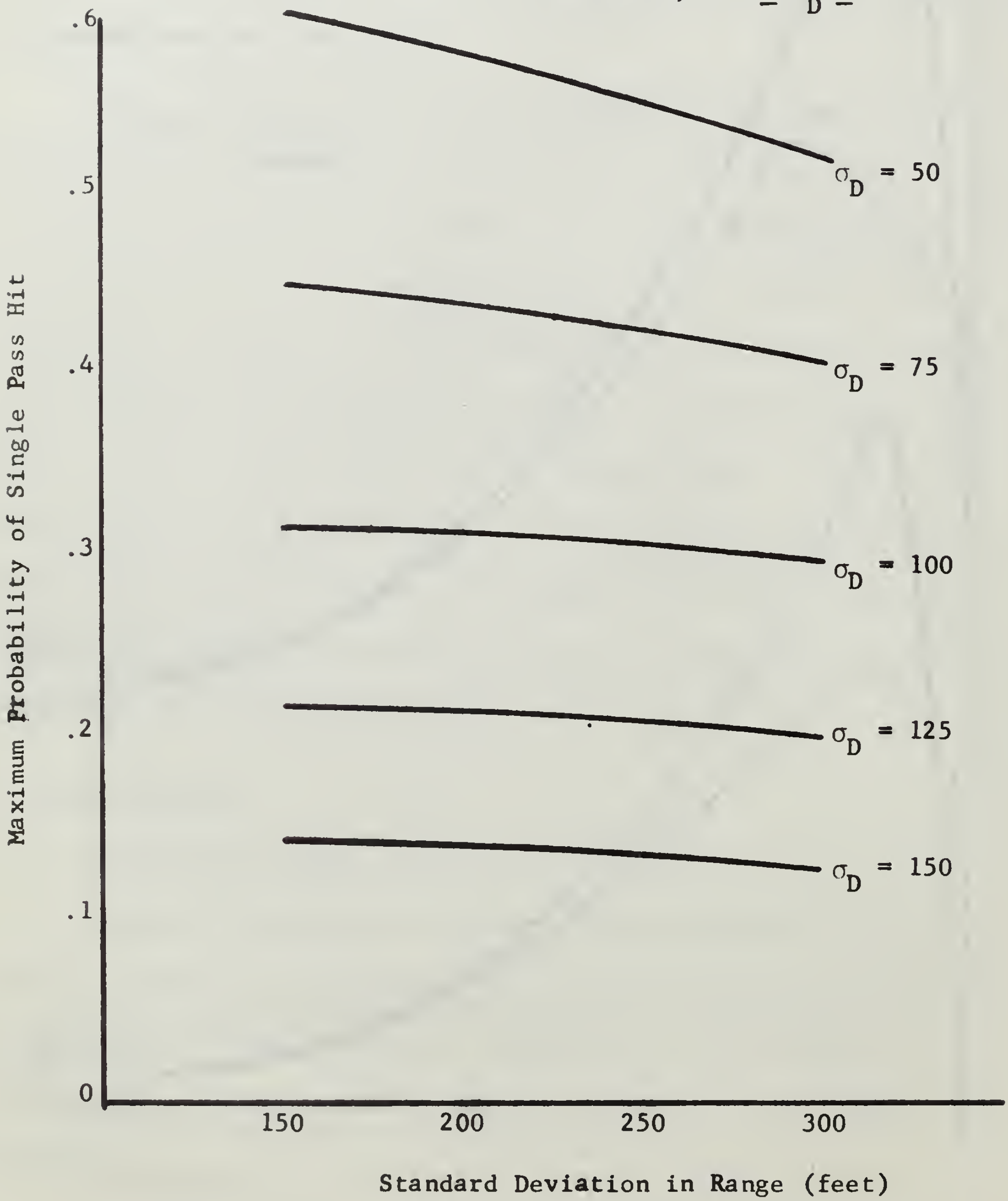
1. Form of the Model

Because of its finite length, the area over which a kill occurs is somewhat different than the infinite strip involved in the railroad problem. In order to examine this area, define an X-Y coordinate system, parallel and perpendicular to the target dimensions.


GRAPH I: Maximum Probability Single Pass Hit vs. Approach Angle
EMD = 12.5'
n = 6 Railroad/Road Target 5' wide
– – – Approach angle error present
σ_D = 50
σ_R = 150
σ_D = 100
σ_R = 300
Maximum Probability Single Pass Hit
.9
.8
.7
.6
.5
.4
.3
.2
.0.1
0
20
40
60
80
90
Approach Angle (degrees)

GRAPH II: Maximum Probability Single Pass Hit vs. Standard Deviation in Range

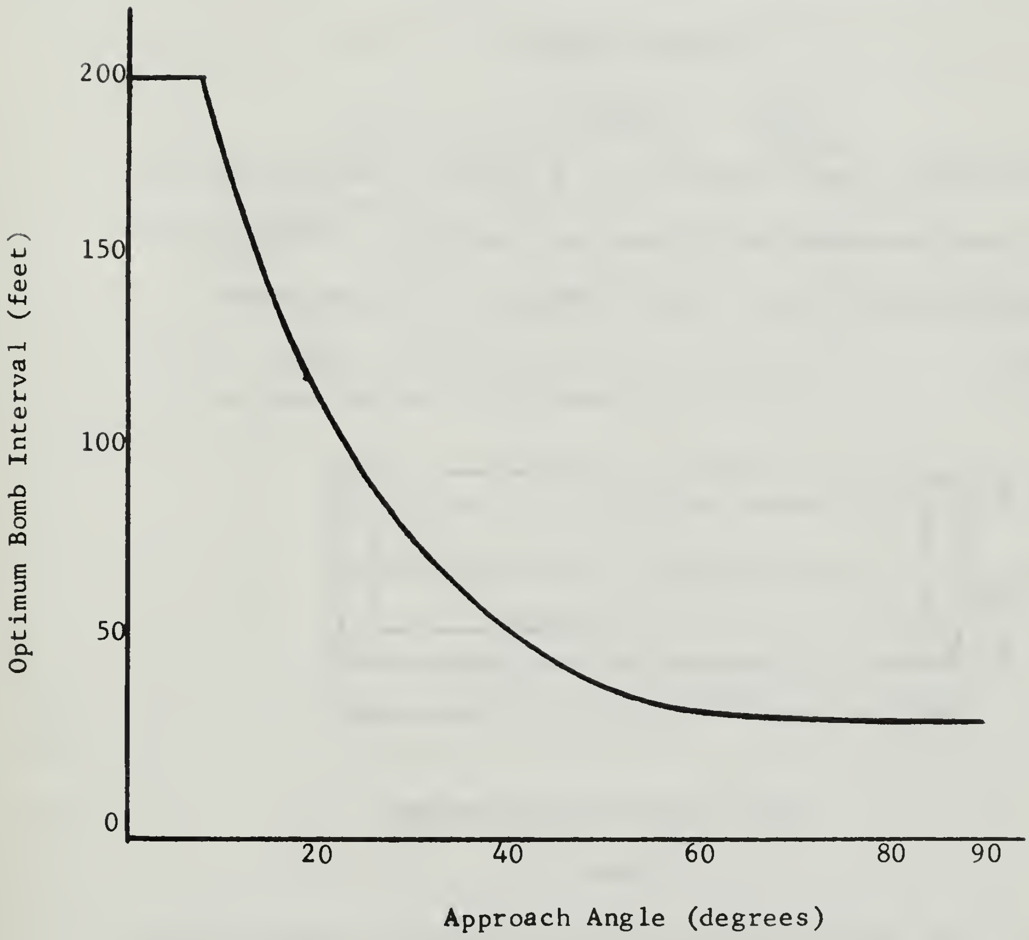
EMD = 12.5'; n = 6; Railroad/Road Target 5' Wide; 50' $\leq \sigma_D \leq$ 150'

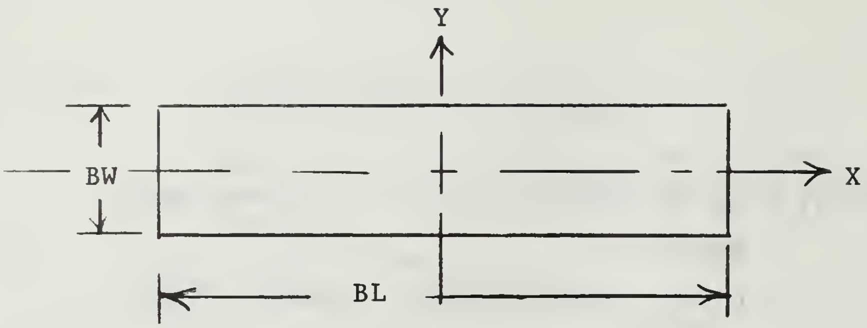


GRAPH III: Optimal Bomb Interval vs. Approach Angle

EMD = 12.5'

n = 6 Railroad/Road Target 5' Wide

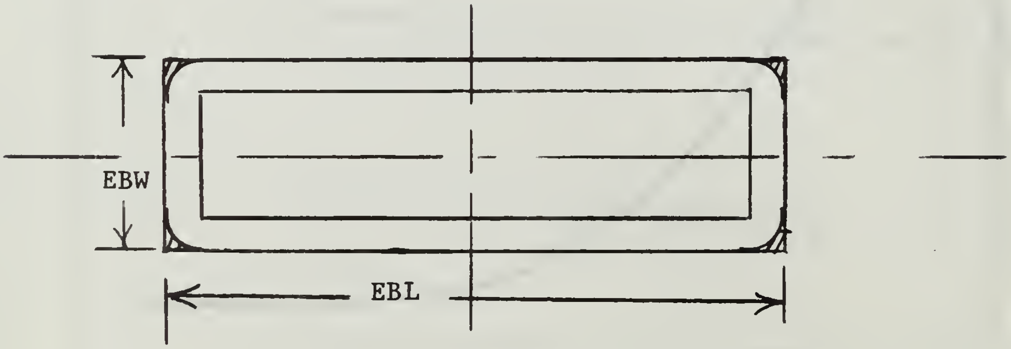




Bridge target

FIGURE 5

Now increase the target dimensions by a quantity representing the effective miss-distance for the weapon in question. Call the new dimensions effective bridge length (EBL) and effective bridge width (EBW).



Effective bridge dimensions

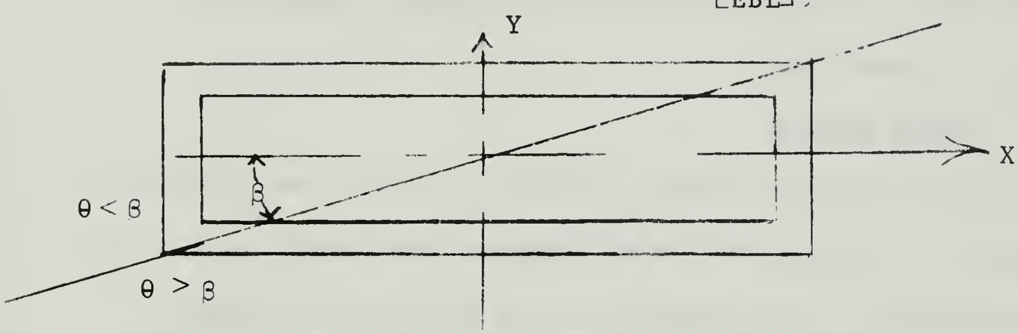
FIGURE 6

Notice that the shaded areas are approximations to the actual distance away from the target, that the centers of impact may be and still produce a kill. The small increase in accuracy obtained by requiring the boundaries to be exact is, however, far outweighed by the complexity introduced in the mathematics involved; hence, the approximation is used throughout the development.

By defining the target in this manner, a target kill is produced whenever the center of impact of a weapon falls within the rectangle defined by EBL and EBW.

The first problem is to recognize that there are two basically different approach angles available in attacking the target: (1) less than the critical angle, and (2) equal to or greater than the critical angle where the critical angle is defined as:

$$\text{Critical angle} = \beta = \tan^{-1}\left[\frac{EBW}{EBL}\right].$$



Critical approach angle

FIGURE 7

Both types of attack are developed but since they differ only in the description of the integration limits only the case when $\theta \geq \beta$ is discussed below. For ease in notation assume n = 2. The optimal bomb interval is determined in exactly the same manner as for the railroad/road problem. That is

$$INT = (2\ EMD + BW)/\sin\theta = EBW/\ \sin\theta.$$

Next, the question of where the center of the stick can fall and still produce a kill must be answered. Once this has been determined and transformed into the R-D coordinate system, see Appendix C, the problem is essentially the same, that is

$$\text{MAX Psph} = \iint_{\substack{\text{proper}\\ \text{polygon}}} f_{\underline{R}\,\underline{D}}(R,D)\, dRdD.$$

Since the distribution of approach angle error produced only minor variations in the results in the railroad problem, in this case $g_{\underline{\theta}}(\theta)$ was assumed to have all of its probability mass at θ_0, hence its effect does not appear in the integral for Psph.

By using the numerical method of Appendix D to evaluate the above integral, the maximum Psph for any target dimensions, σ_R, σ_D, n, θ can be determined.

2. <u>Sample Results</u>

Results are once again presented for only one representative combination of parameters and all comments made apply only to this set of parameters. The parameters used are (1) n = 6 bombs, (2) EMD = 5', (3) 40' $\leq$ bridge length $\leq$ 200', (4) bridge width = 10' and 20', (5) 100' $\leq \sigma_R \leq$ 300', (6) σ_D = 50', 100'.

The graphs are rather self-explanatory but perhaps several facts are worthy of comment. In the first place until a bridge length of approximately 100' is reached, Psph is relatively unaffected by approach angle. Remember, however, that (1) each Psph is computed using the optimum bomb interval for the approach angle in question, and (2) selecting a bomb interval that is not compatible with the approach angle will produce lower results. Once again stick length and bomb interval change rapidly at the shallow angles and are relatively constant for $\theta > 45^\circ$. These two facts point toward approach angles in excess of 45° if large approach angle error or bomb interval error is anticipated.

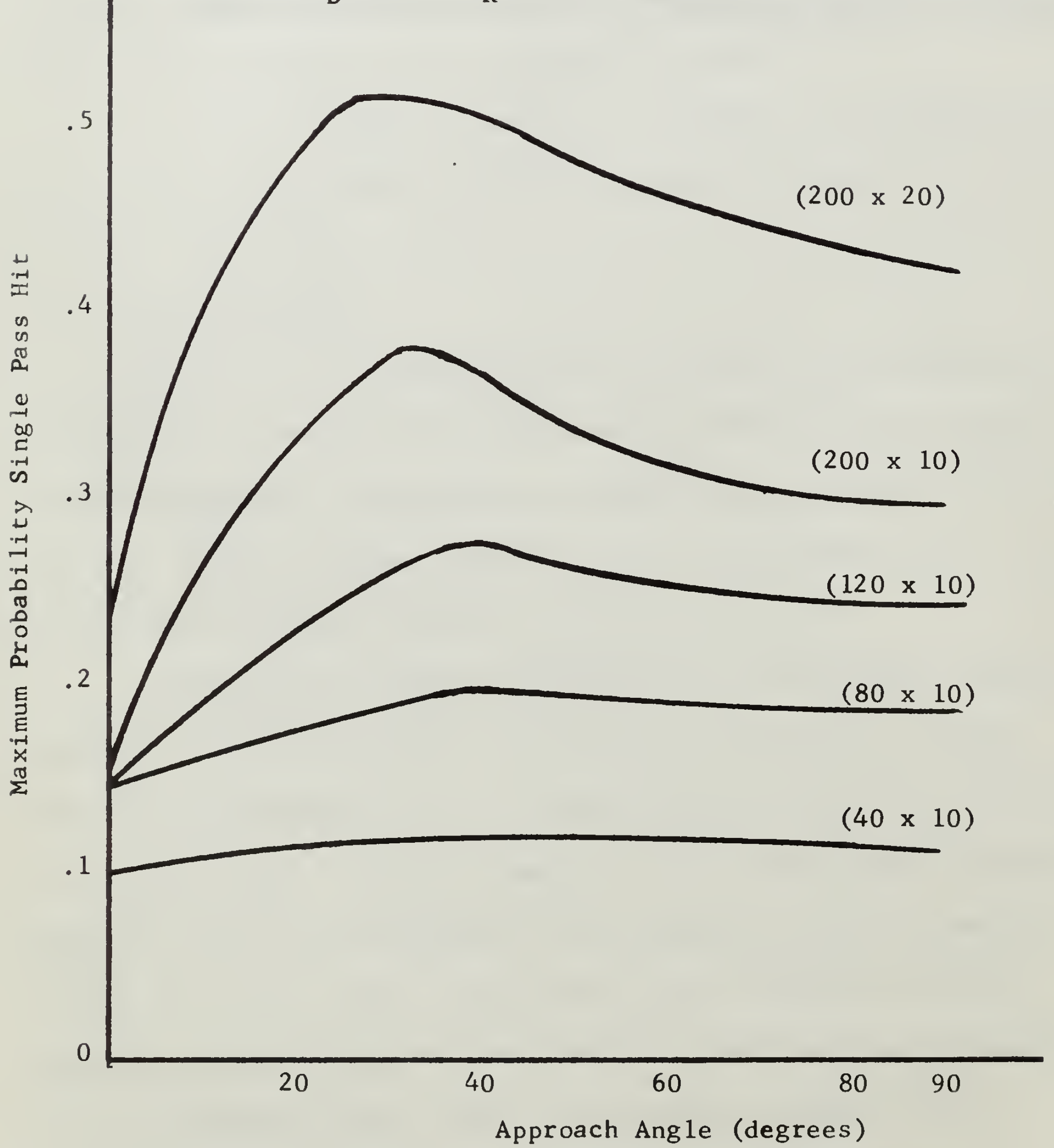
Notice also that as the bridge dimensions increase the difference between the Psph for a pilot miss distribution of σ_R = 150', σ_D = 50' and a pilot miss distribution of σ_R = 300', σ_D = 100' increases. For example when the bridge length is increased from 100' to 200' Psph, for pilot σ_R = 150', σ_D = 50', increases 66%, while Psph for a pilot σ_R = 300', σ_D = 100' increases only 59%.

GRPAH IV: Maximum Probability Single Pass Hit vs. Approach Angle

EMD = 5' n = 6

Bridge Target Width, 10', 20'; 40' $\leq$ length $\leq$ 200'

σ_D = 50'; σ_R = 150' (Length x width)

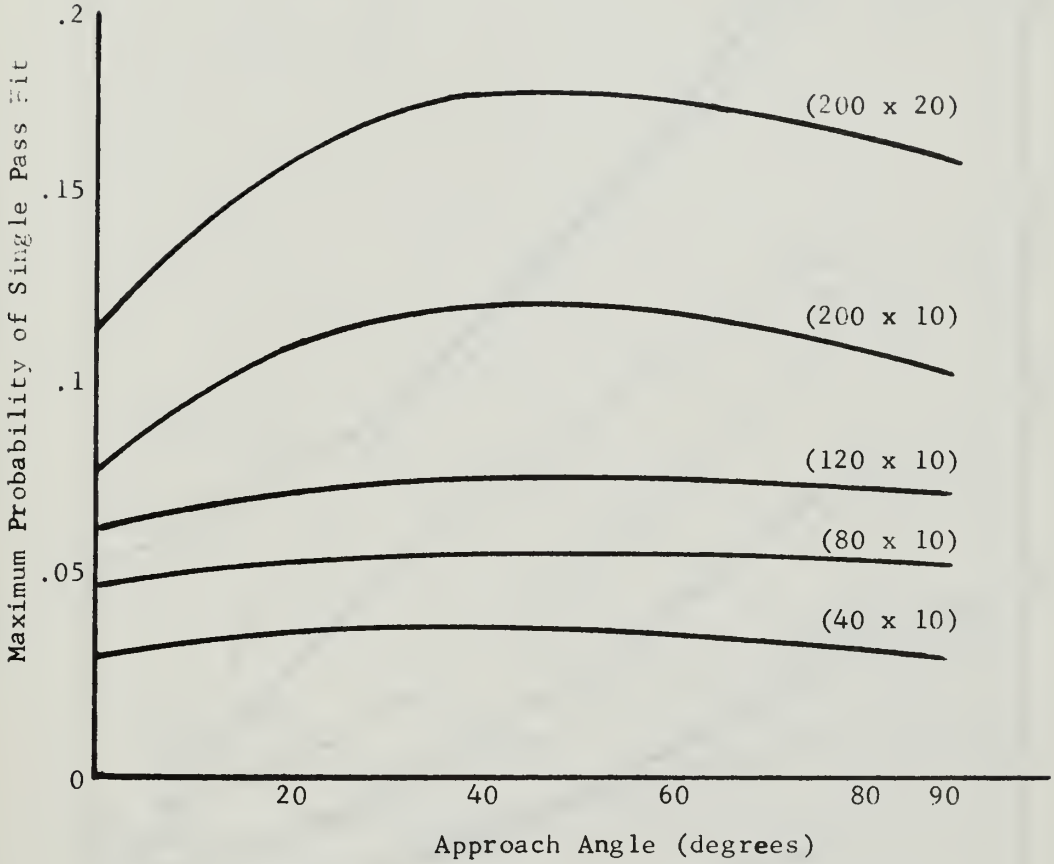


GRAPH V: Maximum Probability of Single Pass Hit vs. Approach Angle

EMD = 5' n = 6

Bridge Width 10', 20'; 40' ≤ Length ≤ 200'
(Length x width)

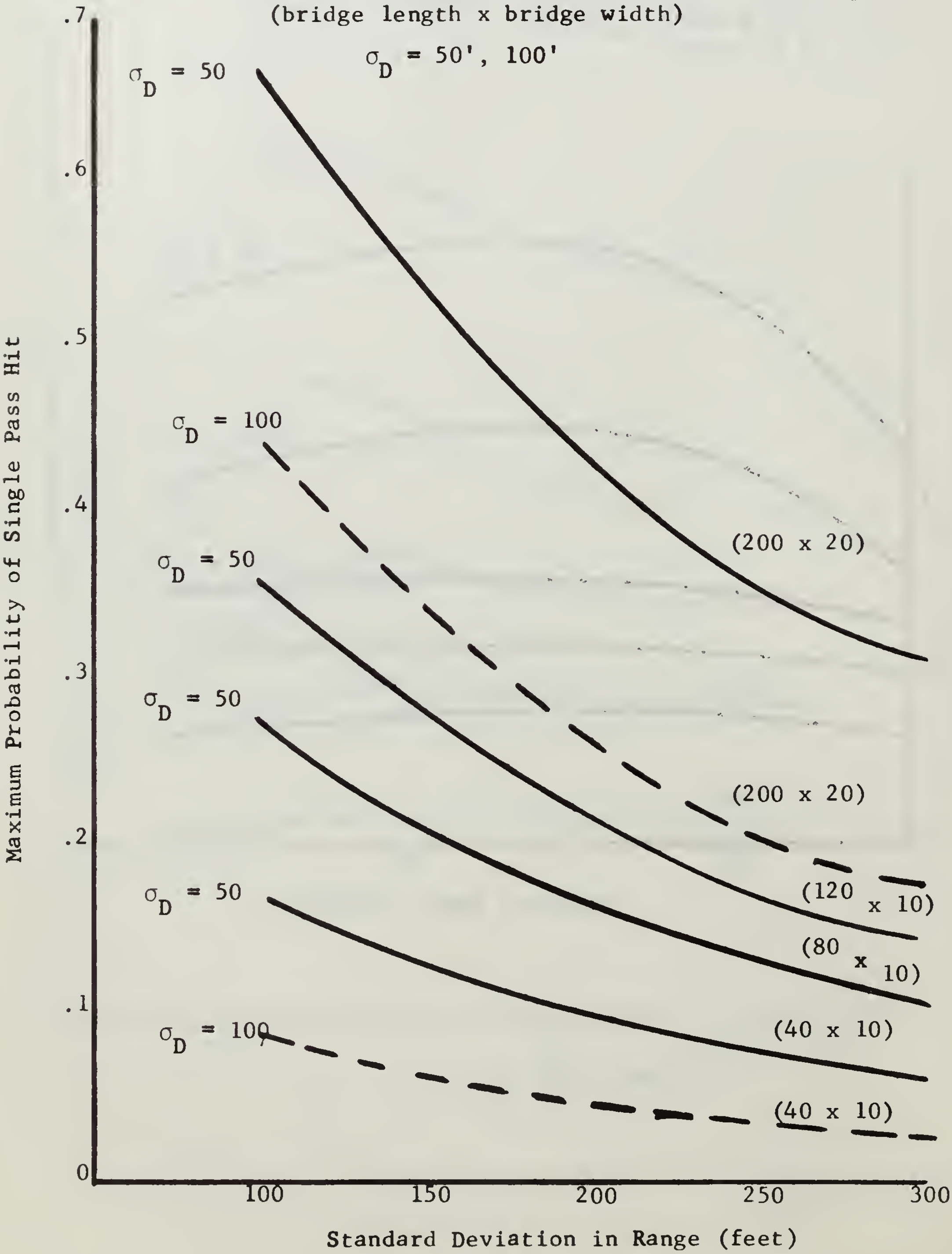
σ_D = 100', σ_R = 300'



GRAPH VI: Maximum Probability Single Pass Hit vs. Standard Deviation in Range

EMD = 5' n = 6

Bridge Target width 10', 20'; 40' < Length ≤ 200'
(bridge length x bridge width)

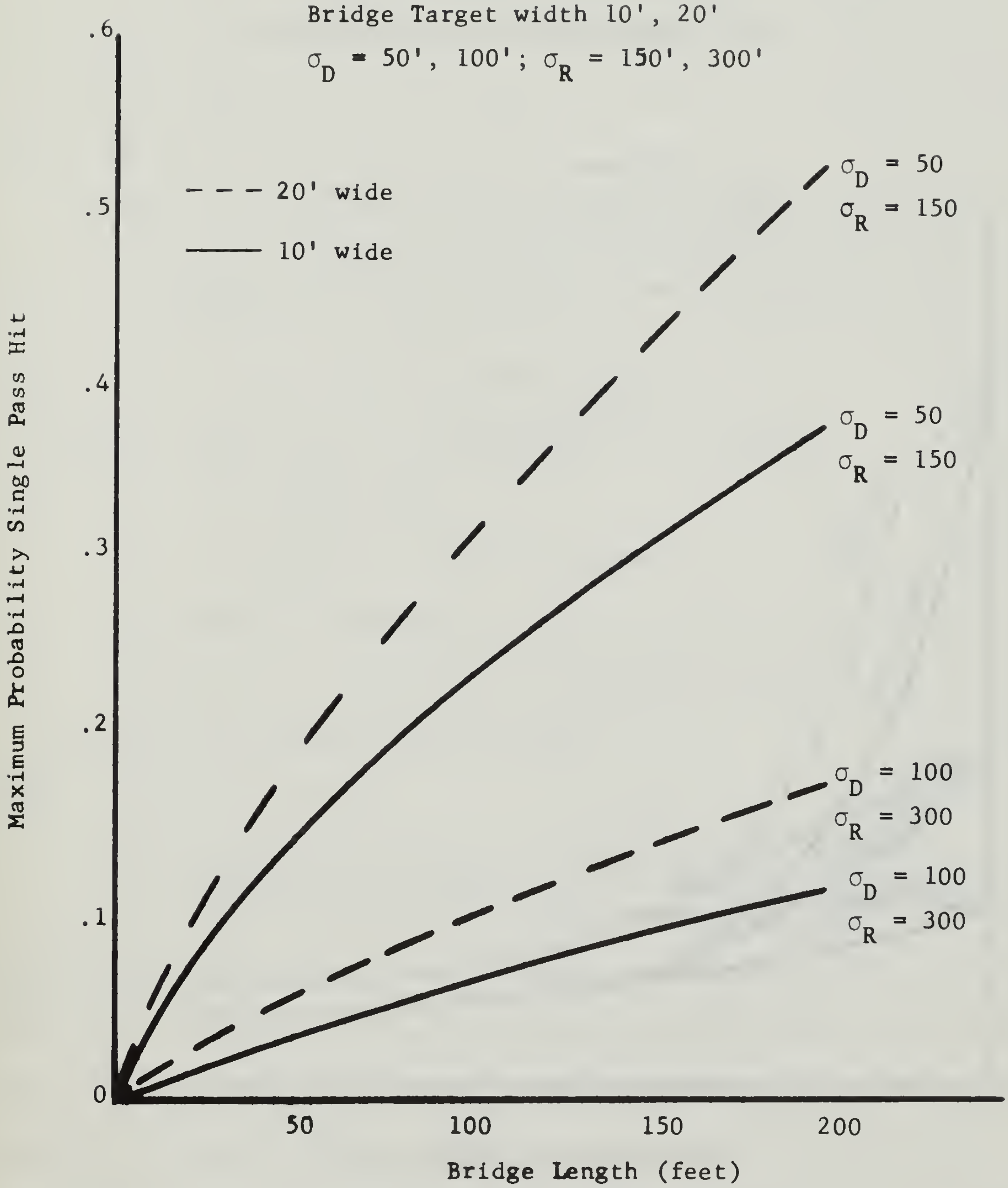


GRAPH VII: Maximum Probability Single Pass Hit vs. Bridge Length

EMD = 5' $\qquad$ n = 6

Bridge Target width 10', 20'

σ_D = 50', 100'; σ_R = 150', 300'

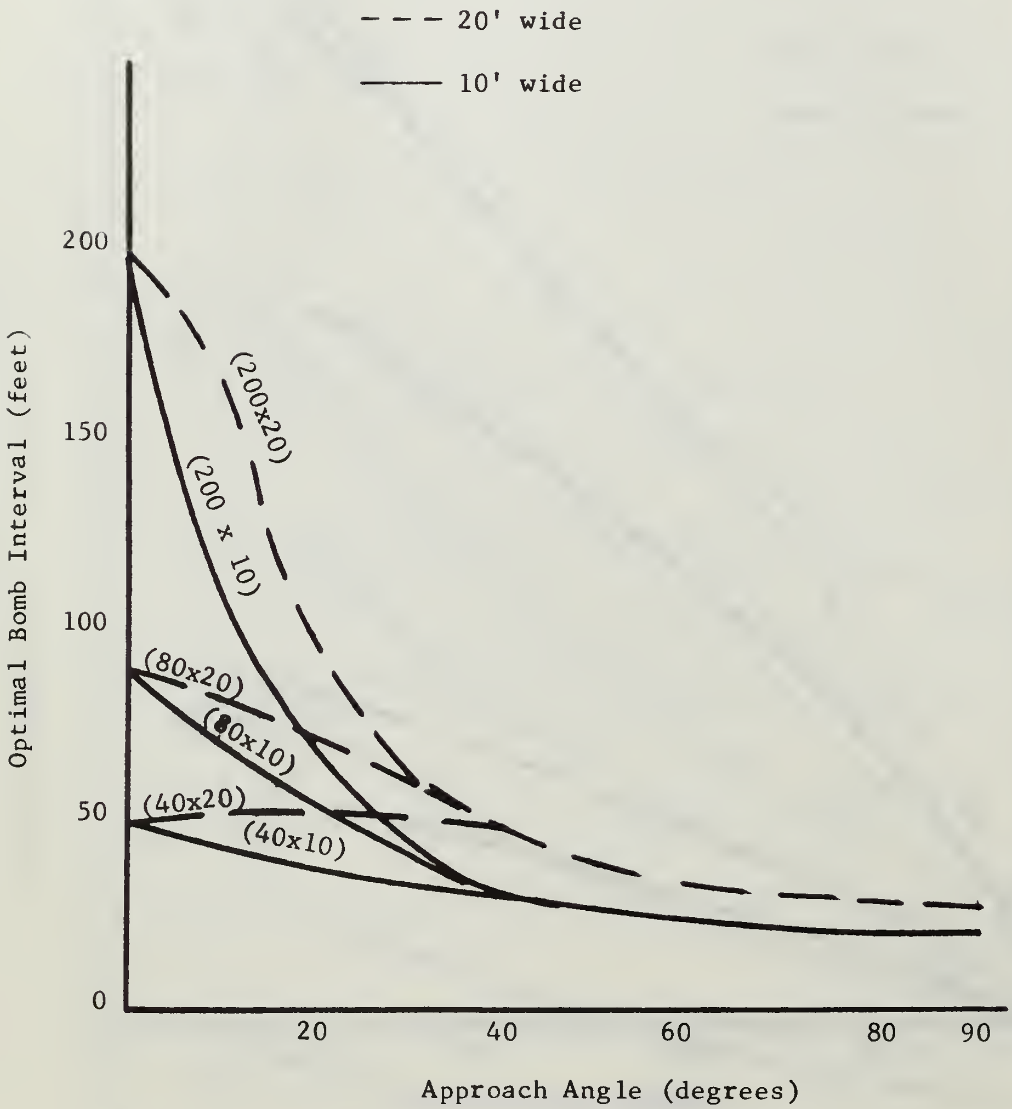


GRAPH VIII: Optimal Bomb Interval vs. Approach Angle

EMD = 5' n = 6

Bridge Target width 10', 20'

40 ≤ length ≤ 200' (length x width)



IV. CONCLUSIONS

In conclusion it is evident that for any given target dimensions, bomb size, number of bombs, and pilot miss-distance distribution, there is definitely an approach angle and bomb interval that maximizes the probability of a single pass hit.

Furthermore it has been shown that for all targets considered:

(1) the optimal bomb interval is independent of the pilot miss-distance distribution and can be computed by the formula:

$$INT = [2\ EMD + W]/\sin\theta$$

where EMD ≡ effective miss-distance of the weapon

W ≡ target width

θ ≡ preselected approach angle,

(2) optimal stick length

$$SL = (n-1)\ [2\ EMD + W]/\sin\theta$$

where n = number of bombs dropped,

and

(3) although, in general, no closed form exists for the optimal approach angle, for all cases investigated the optimal angle was in the neighborhood of 10^{o}-30^{o}.

This latter angle, however, is critical, since Psph is quite sensitive to it. For example in the railroad target problem, while approaching the target from approximately 10^{o} produced a maximum Psph, approaching from 0^{o} produced a minimum. Likewise for the bridge target problem, approaching the target from an angle slightly greater than the diagonal produced a maximum Psph, but when the approach angle became less than the diagonal, the Psph was greatly reduced.

It was also shown that the rate of change of the optimal bomb interval was much larger for small approach angles than for approach angles greater than 45^o. In fact for approach angles greater than 45^o, the optimal bomb interval was almost constant. These facts indicate that it may be better to use a non-optimal approach angle greater than 45^o if large approach angle/bomb interval errors are anticipated.

For the railroad/road problem, it was shown that the optimal approach angle also depends on the maximum and minimum release altitudes and the dive angle (α) by the relationship

$$\theta_{opt} = \sin^{-1}\left[\frac{(ALT_{max} - ALT_{min}) \cot(\alpha)}{(n-1)\ (2\ EMD + RRW)}\right]$$

where α = dive angle,

RRW = railroad width.

Once again, however, approach angle is independent of pilot miss-distance distribution. Furthermore for the cases investigated, Psph was relatively constant for all values of σ_R, when σ_D was greater than 50'. This indicates that effort should first be expended to reduce the error in the deflection direction if σ_D is greater than 50'. Lastly, it was shown that if approach angle error was introduced, only minor variations of the results were produced, and for $\theta > 45^o$, this variation was negligible.

In the bridge problem, it was shown that for bridges less than 100 feet in length, Psph is not sensitive to approach angle as long as the optimal bomb interval is used for the preselected approach angle. As the bridge length increased the results asymptotically approached those of the railroad problem. It should be noted,

however, that it was impossible to calculate the optimal approach angle prior to performing the numerical integration, but in all cases the optimal approach angle was slightly greater than the angle formed by the diagonal.

One last comment must be made about the results derived from the models and the conclusions drawn therefrom. It has been assumed that a pilot will use the optimal bomb interval for the preselected approach angle or expected approach angle. Either the derived results and/or the conclusions drawn therefrom may cease to be applicable if the bomb interval and approach angle do not satisfy the equation:

$$\mathrm{INT} = [2\ \mathrm{EMD} + \mathrm{W}]/\sin\theta_0.$$

APPENDIX A

OPTIMAL BOMB INTERVAL

Before the formulation of an optimal bomb interval can be developed, some discussion as to how stick bombing affects pilot error is in order. Stick bombing is based upon the fact that during the tracking dive, the pilot is able to release his weapons at programmed intervals. These weapons, at least theoretically, fall in a "straight line" (as modified by gravity) and, except for wind and ballistic effects, directly along the flight path. Implicit in this assumption is the fact that any wind or ballistic error affect all bombs uniformly and thus do not disturb the "straight line" effect of the tactic. Furthermore, since during the tracking run the pilot attempts to maintain an essentially straight line dive at least during the release of the weapons, any deflection error will affect all weapons uniformly. It is evident then that stick bombing does not affect deflection error but does have an affect on range error. In other words, by changing the stick length (i.e., the bomb interval), for any given pilot miss-distance distribution the probability of a single pass hit (Psph) changes, and this change depends upon the relationship between σ_R, stick length (SL), bomb interval (INT), and of course target size and bomb effectiveness (EMD).

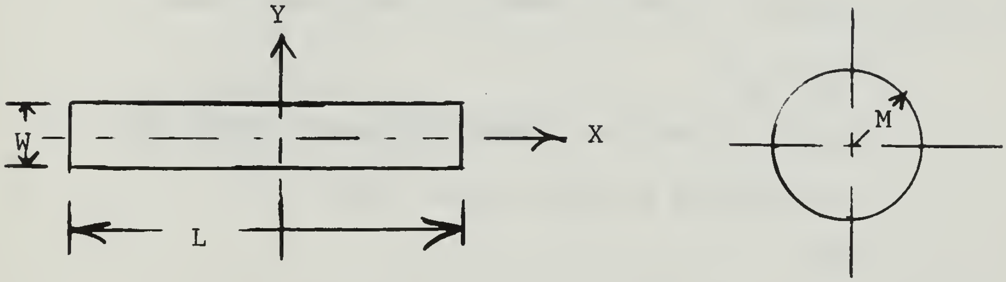
Although bomb interval and σ_R are needed in order to calculate Psph, it is possible to show that development of the optimal bomb interval (criteria: maximize Psph) does not in fact depend on σ_R

but only on the target dimensions, bomb type (i.e., Effective Miss Distance, EMD) and approach angle.

From the definitions of Stick Length (SL) and Bomb Interval (INT), it is clear that

$$SL = INT\ (n-1)$$

where n = number of bombs dropped. Now suppose that the target (either railroad, road, or bridge) has width (W) and length (L), and that the EMD of each weapon is M feet. See Figure 8.

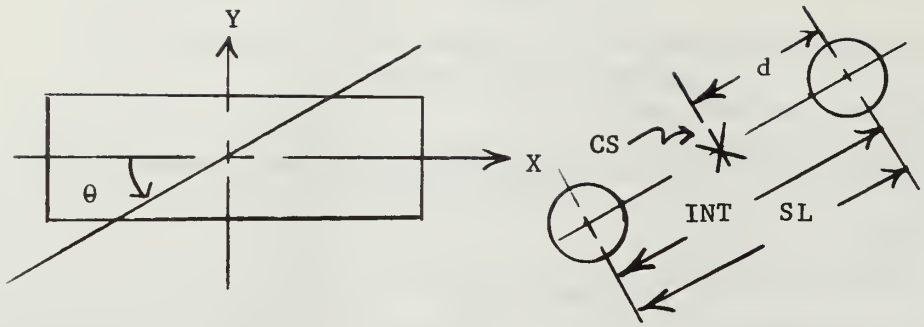


Target and EMD of bomb

FIGURE 8

Assume furthermore that the approach angle (θ) is known. This assumption can be viewed in two different ways: (1) it applies if the pilot preselects his approach angle and makes his approach at that angle without error or, (2) it applies if the pilot preselects his approach angle and attempts to approach the target at that angle but has an error of approach angle (EAA).

Now let us assume that the number of bombs in the stick is 2, (n = 2). Once the development for two bombs has been completed the results can readily be extended to sticks of 2, 3, ..., n bombs. See Figure 9.



Angle of approach and stick

FIGURE 9

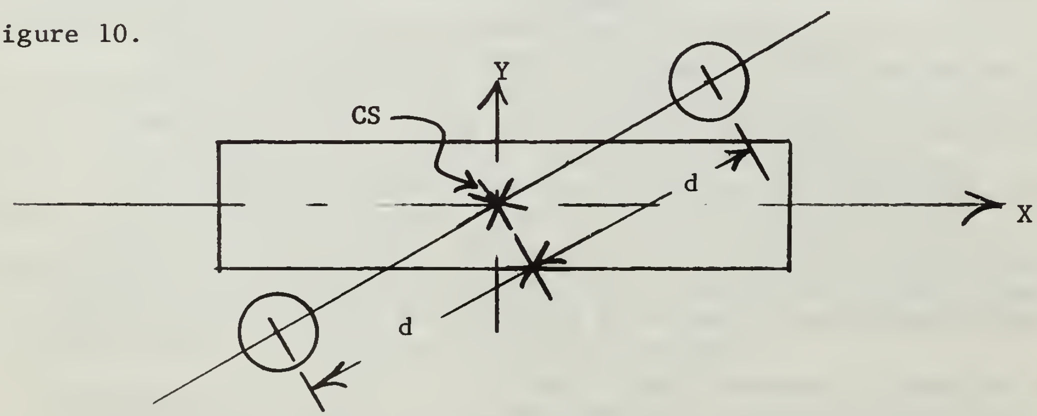
Define CS = geometric center of the stick with coordinates (X_{CS}, Y_{CS})

d = distance from CS to Center of Impact (CI) of the first weapons on either side of CS

Clearly 2d = INT

and (n-1)(INT) = SL = INT.

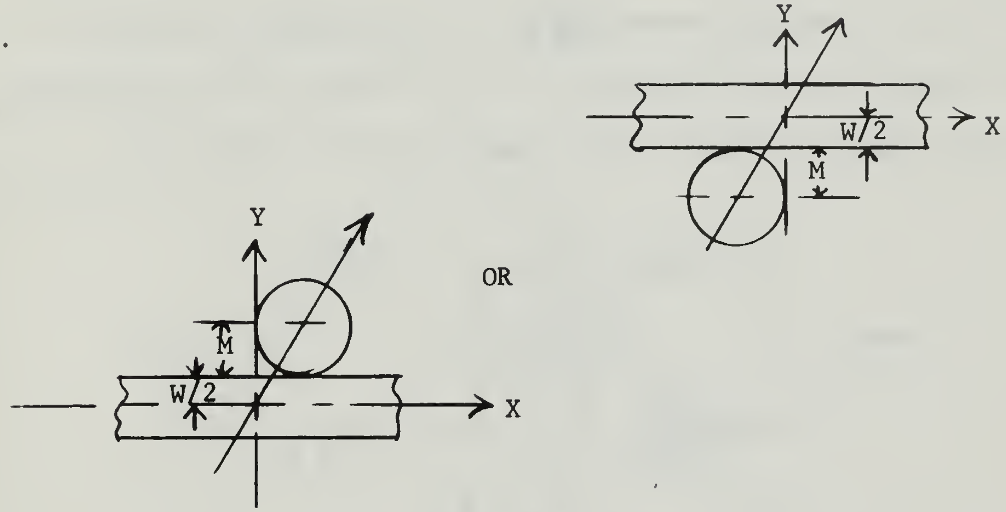
Now suppose that CS falls upon the center line of the target. See Figure 10.



Bomb stick superimposed on target

FIGURE 10

Define a X-Y coordinate system along and perpendicular to the center line of the target. Then a necessary condition for a kill is to require the CI of either weapon to be within $\pm$ [EMD + W/2] (as measured in the Y-direction) of the target center line. See Figure 11.



CI limits needed to produce a target kill

FIGURE 11

It is not a sufficient condition because the error in the X-direction may be so large as to place the entire stick off the target area. This fact, however, does not affect the development of the optimal stick length because the expected aim point is the geometric center of the target area.

Define: $b = M + W/2$,

then the necessary condition for a kill is

$$-b \leq [Y_{CS} + d] \sin \theta \leq b$$

or

$$-b \leq [Y_{CS} - d] \sin \theta \leq b.$$

The first inequality produces

$$-\frac{b}{\sin \theta} - d \leq Y_{CS} \leq \frac{b}{\sin \theta} - d,$$

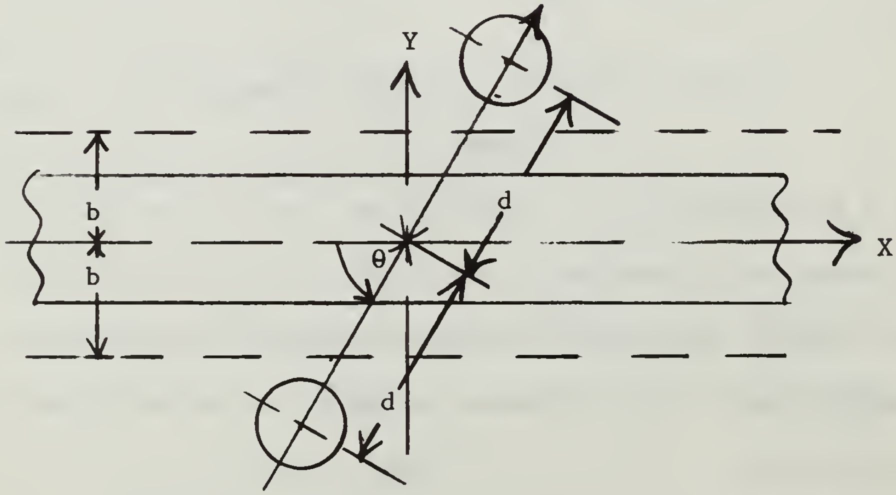
and the second inequality produces

$$-\frac{b}{\sin\theta} + d \leq Y_{CS} \leq \frac{b}{\sin\theta} + d.$$

There are two possible conditions to investigate, $\frac{b}{\sin\theta} < d$, and $\frac{b}{\sin\theta} > d$.

In the first place, suppose $\frac{b}{\sin\theta} < d$. See Figure 12. For this case two bombs never simultaneously hit the target and the probability of a single pass hit (Psph) in the range direction is given by

$$\text{Psph} = \int_{-d-\frac{b}{\sin\theta}}^{-d+\frac{b}{\sin\theta}} f_{\underline{R}}(R)\,dR + \int_{d-\frac{b}{\sin\theta}}^{d+\frac{b}{\sin\theta}} f_{\underline{R}}(R)\,dR.$$



Stick, b/sin θ < d

FIGURE 12

Since $f_{\underline{R}}(R) \sim \text{NORMAL}(0, \sigma_R)$, normalizing this expression produces

$$\text{Psph} = \Phi\left(-\frac{d+b/\sin\theta}{\sigma_R}\right) - \Phi\left(-\frac{d-b/\sin\theta}{\sigma_R}\right) + \Phi\left(\frac{d+b/\sin\theta}{\sigma_R}\right) - \Phi\left(\frac{d-b/\sin\theta}{\sigma_R}\right)$$

which because of the symmetry of the normal distribution reduces to

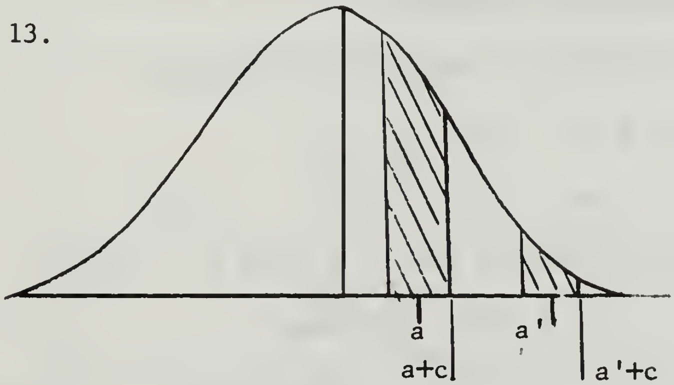
$$Psph = 2\left[\Phi\left(\frac{d+b/\sin\theta}{\sigma_R}\right) - \Phi\left(\frac{d-b/\sin\theta}{\sigma_R}\right)\right].$$

Now suppose d increases, then Psph decreases because

$$\Phi\left(\frac{a+c}{\sigma}\right) - \Phi\left(\frac{a-c}{\sigma}\right) > \Phi\left(\frac{a'+c}{\sigma}\right) - \Phi\left(\frac{a'-c}{\sigma}\right)$$

when a' > a.

See Figure 13.

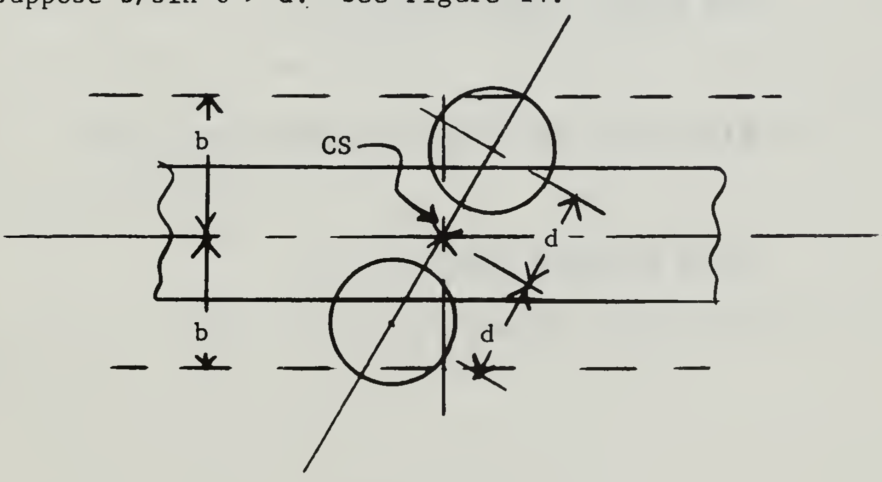


Normal density function

FIGURE 13

$\therefore$ When $\dfrac{b}{\sin\theta} < d$, the Psph decreases as d increases.

Now suppose $b/\sin\theta > d$. See Figure 14.



Stick, $b/\sin\theta$ d

FIGURE 14

Now

$$Psph = \int_{-d - b/\sin\theta}^{d + b/\sin\theta} f_{\underline{R}}(R)\, dR$$

and by normalizing

$$Psph = 2\left[\Phi\left(\frac{d + b/\sin\theta}{\sigma_R}\right) - \Phi(0)\right].$$

Clearly as d increases Psph increases.

$\therefore$ When $b/\sin\theta > d$, Psph increases as d increases.

Since when $b/\sin\theta > d$, Psph $\uparrow$ as d $\uparrow$

and when $b/\sin\theta < d$, Psph $\downarrow$ as d $\uparrow$,

the Psph must reach its maximum when $d = b/\sin\theta$, or when

$$d = [EMD + W/2]/\sin\theta.$$

This argument is readily extended to sticks containing more than two weapons by arguing that each interval should be set so as to produce the maximum Psph, hence each interval between weapons should be set so that

$$INT = 2d = 2[EMD + W/2]/\sin\theta$$

and

$$SL = (n-1)\ INT = 2(n-1)\ [EMD + W/2]/\sin\theta$$

or

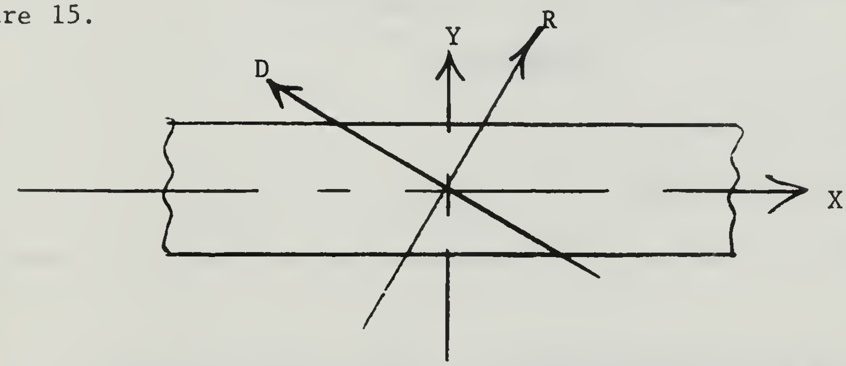
$$INT = [2\ EMD + W]/\sin\theta$$

$$SL = (n-1)\ (2\ EMD + W)/\sin\theta.$$

<u>APPENDIX B</u>

LIMITS OF INTEGRATION RAILROAD/ROAD TARGETS

In order to discuss the limits of integration define two coordinate systems, a X-Y coordinate axis parallel and perpendicular to the target dimensions and a R-D coordinate system (used to measure pilot error) parallel and perpendicular to the flight path. See Figure 15.



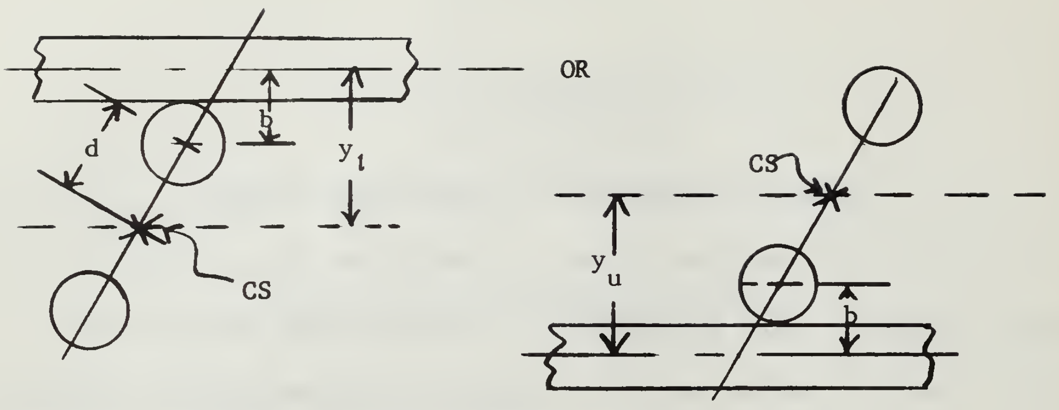
Coordinate axes in railroad/road problem

FIGURE 15

Once again in order to reduce the notation assume that n = 2 (i.e., the stick consists of two weapons). Furthermore, for any given approach angle (θ), set the bomb interval (INT) for these weapons at the optimal distance,

$$INT = (2\ EMD + RRW)/\sin\theta.$$

Now, there is a target kill if the CS falls within certain limits. See Figure 16.



Limits of position of stick center
needed to produce a target kill

FIGURE 16

where y_u = upper limit for CS and

y_l = lower limit for CS.

The question then is what is the mathematical representation of these limits.

From Figure 16 it is apparent that $y_u = b + d \sin \theta$ and that $y_l = -y_u$. From Appendix A

$$b = (EMD + RRW/2)$$

$$d = (EMD + RRW/2)/\sin \theta$$

hence

$$y_u = (EMD + RRW/2) + (EMD + RRW/2) \sin \theta/\sin \theta$$

$$= 2\ EMD + RRW$$

and

$$y_l = -\ [2\ EMD + RRW];$$

however, the pilot miss-distance is measured in the R-D coordinate system and must be rotated into the X-Y coordinate system before the integration can take place. Therefore, introduce a change of variable

$$R = X \cos \theta + Y \sin \theta$$

$$D = -X \sin \theta + Y \cos \theta.$$

The Jacobian of this transformation is

$$\left|\frac{\partial(R,D)}{\partial(X,Y)}\right| = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = 1$$

and the

$$Psph = \int_{-\infty}^{\infty} \int_{-[2\ EMD+RRW]}^{[2\ EMD+RRW]} f_{\underline{X}\ \underline{Y}}(X,Y)\ dXdY$$

where

$$f_{\underline{X}\ \underline{Y}}(X,Y) = \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\ \overline{XY}\ A \begin{pmatrix} X \\ Y \end{pmatrix}\right]$$

and

$$A = \begin{pmatrix} \frac{\cos^2\theta}{\sigma_R^2} + \frac{\sin^2\theta}{\sigma_D^2}, & \frac{-\cos\theta\ \sin\theta}{\sigma_R^2} + \frac{\cos\theta\ \sin\theta}{\sigma_D^2} \\ -\frac{\cos\theta\ \sin\theta}{\sigma_R^2} + \frac{\cos\theta\ \sin\theta}{\sigma_D^2}, & \frac{\sin^2\theta}{\sigma_R^2} + \frac{\cos^2\theta}{\sigma_R^2} \end{pmatrix}.$$

The double integral, although having limits of integration that are easy to evaluate, has an integrand that precludes a closed form evaluation. However, this integral does provide some insight into the problem. Notice that the limits of integration are independent of the approach angle (θ). This, of course, means that no matter what value θ takes on, the infinite strip over which the integration will take place will be the same. Furthermore, if this infinite strip is represented by coordinates in the R-D coordinate system, it will remain unchanged as far as size and shape is concerned. These facts are true because the transformation involved is simply an area

preserving rotation of axis. This can be seen by evaluating the Jacobian of the transformation

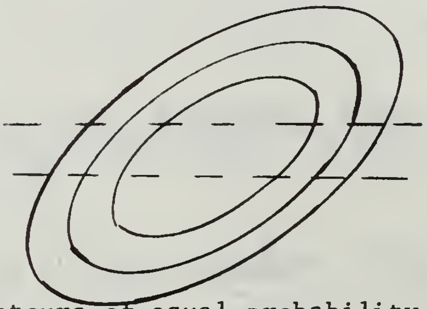
$$X = R \cos \theta + D \sin \theta$$

$$Y = -R \sin \theta + D \cos \theta$$

$$\left|\frac{\partial(X,Y)}{\partial(R,D)}\right| = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = 1$$

and recalling that if the absolute value of the Jacobian of a transformation equals 1, then the transformation does not change the scale of any dimensions.[4]

The problem now then is to find for a given bi-variate normal distribution the orientation of a fixed dimension infinite strip so as to sweep out the maximum probability. See Figure 17.



Contours of equal probability of a bi-variate normal distribution, $\sigma_R > \sigma_D$

FIGURE 17

Obviously, the maximum probability is swept out if the infinite strip lies along the major axis of the ellispes of equal probability with its center line coinciding with the R-axis. See Figure 18.

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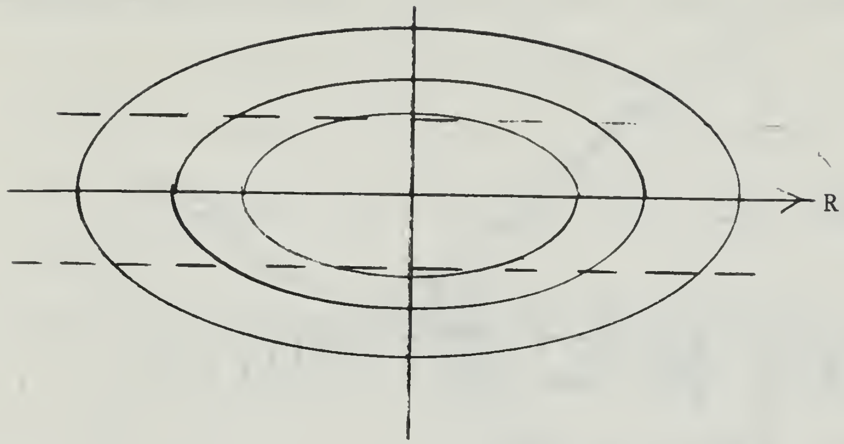
[4] Kaplan, Wilfred, <u>Advanced Calculus</u>, p. 201, Addison Wesley, 1957.



Infinite strip sweeping out maximum probability
for a given bi-variate normal distribution

FIGURE 18

However, when $\theta = 0^{o}$ is used to evaluate the bomb interval where

$$INT = [2\ EMD + RRW]/\sin\theta$$

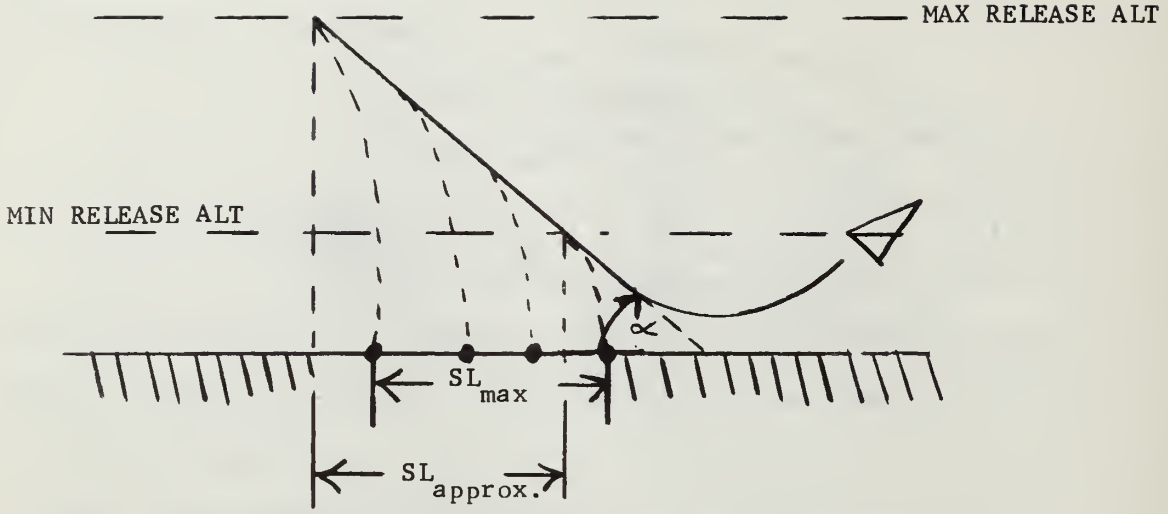
the interval $\longrightarrow \infty$.

The meaning of this result is that the interval should be infinitely long. This, of course, is impossible to achieve but it does point out that the bomb interval should be as great as possible, while still satisfying the conditions for interval optimality. Recall that there has been defined a maximum and minimum release altitudes. These two constraints limit the bomb interval. See Figure 19.

Clearly $SL_{max} \approx [ALT_{max} - ALT_{min}]\ \text{cotan}\ \alpha$ where α = dive angle. and since

$$INT = SL/(n-1)$$

$$INT_{max} \approx \frac{(ALT_{max} - ALT_{min})\ \text{cotan}\ \alpha}{(n-1)}.$$



Maximum stick length, n = 4

FIGURE 19

When n = 2 this expression reduces to

$$INT_{max} = (ALT_{max} - ALT_{min}) \text{ cotan } \alpha.$$

Now the interval must also satisfy the expression

$$INT_{max} = [2 \text{ EMD} + \text{RRW}]/\sin\theta$$

in order to produce an optimal stick length. Hence it is possible to determine the best possible approach angle:

$$\frac{(ALT_{max} - ALT_{min}) \text{ cotan } \alpha}{n-1} = INT_{max} = (2 \text{ EMD} + \text{RRW})/\sin\theta$$

or

$$\frac{1}{\sin\theta} = \frac{(ALT_{max} - ALT_{min}) \text{ cotan } \alpha}{(n-1)\ (2 \text{ EMD} + \text{RRW})}$$

or

$$\theta_{opt} = \sin^{-1}\left\{\frac{1}{\frac{[ALT_{max} - ALT_{min}] \text{ cotan } \alpha}{(n-1)\ (2 \text{ EMD} + \text{RRW})}}\right\}.$$

In order to provide specific results and to satisfy the requirements of the computer programs the value for $\left[(ALT_{max} - ALT_{min}) \cot \alpha\right]$ was arbitrarily set equal to 1000 feet. This corresponds to a maximum and minimum release altitude difference of approximately 1000' if the dive angle = 45^{o}.

Although it is now possible to determine what the best approach angle is, it is not possible to determine what Psph is for any specific value of θ nor is it possible to evaluate how much the Psph is degraded by varying from the optimal conditions. To achieve these results it is necessary to evaluate Psph for all θ.

Since rotating the pilot miss-distance distribution coordinate system into the target coordinate system produced a difficult integrand, the reverse procedure might produce better results. That is, rotate the limits of integration into the R-D coordinate system. Recall that the integrals involved are

$$Psph = \int_{-\infty}^{\infty} \int_{-[2EMD+RRW]}^{2EMD+RRW} f_{\underline{X}\,\underline{Y}}(X,Y)\,dY\,dX$$

and

$$Psph = \iint_{\substack{\text{appropriate} \\ \text{strip}}} f_{\underline{R}\,\underline{D}}(R,D)\,dR\,dD$$

where

$$f_{\underline{X}\,\underline{Y}}(X,Y) = \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\,\overline{XY}\,A\begin{pmatrix}X\\Y\end{pmatrix}\right]$$

and

$$f_{\underline{R}\,\underline{D}}(R,D) = \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right].$$

consider the transformation

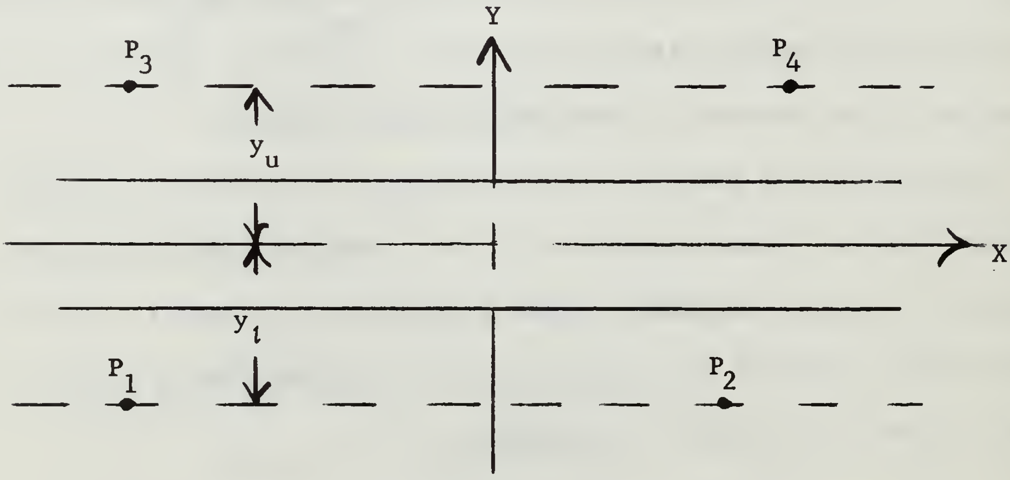
$$X = R \cos \theta + D \sin \theta$$

$$Y = R \sin \theta + D \cos \theta$$

once again

$$\frac{\partial(X,Y)}{\partial(R,D)} = 1.$$

The problem then is to use this transformation on the limits of integration. Recall what the area of integration under consideration looked like in the X-Y coordinate system. See Figure 20.



Area of integration in X-Y coordinate system

FIGURE 20

Since the transformation used is area preserving all that is necessary is to represent this infinite strip in the R-D coordinate system.

Define points

$$P_1 = (X_1, Y_1)$$
$$P_2 = (X_2, Y_2)$$
$$P_3 = (X_3, Y_3)$$
$$P_4 = (X_4, Y_4).$$

Then use the transformation

$$P_1\ (X_1, Y_1) \rightarrow P_1\ (R_1, D_1)$$

$$P_2\ (X_2, Y_2) \rightarrow P_2\ (R_2, D_2)$$

$$\begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

$$P_4\ (X_4, Y_4) \rightarrow P_4\ (R_4, D_4).$$

Now since $(P_2 - P_1)$ defined a line in the X-Y coordinate system, it defines a line in the R-D coordinate system and the equation for this line has the form:

$$D_R = m_1 R + b_1$$

where the subscript R refers to the right hand line in Figure 21, and

$$m_1 = \text{slope} = \frac{R_2 - R_1}{D_2 - D_1}$$

$$b_1 = \text{intercept} = R_1 - \left(\frac{R_2 - R_1}{D_2 - D_1}\right) D_1 .^{5}$$

Likewise the points $(P_4 - P_3)$ define a similar line

$$D_L = m_2 R + b_2$$

where

$$m_2 = \frac{R_4 - R_3}{D_4 - D_3}$$

$$b_2 = R_3 - \left(\frac{R_4 - R_3}{D_4 - D_3}\right) D_3 .$$

These equations produce an area of integration of the form seen in Figure 21.

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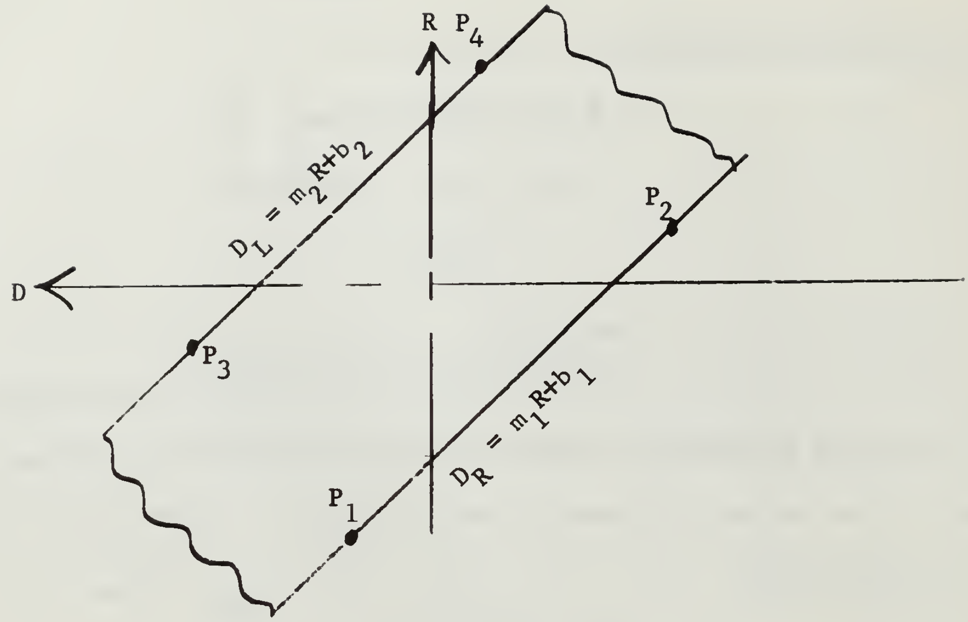
[5] Sisam, C. H., College Mathematics, p. 185, Holt, 1957.



Area of integration in R-D coordinate system

FIGURE 21

Then

$$Psph = \int_{-\infty}^{\infty} \int_{D_R = m_1R+b_1}^{D_L = m_2R+b_2} \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right] dDdR.$$

This integral is easily evaluated on a computer by a numerical integration technique (see Appendix D) allowing R to range from $-\infty$ to $+\infty$.

It is now possible to evaluate Psph at any approach angle (θ), determine the optimal approach angle, evaluate the optimal bomb interval and furthermore, determine the effects on Psph of changing from the optimal tactics.

Although this development was performed when n = 2 it is easily extended to n = 3, 4, ..., n. The only portions of the development

that change are the formulae for determining y_u and y_l. Recall in the case n = 2

$$y_u = b + d \sin \theta$$

$$y_l = -[b + d \sin \theta]$$

and

$$b = EMD + RRW/2$$

$$d = [EMD + RRW/2]/\sin \theta.$$

For a stick of greater than n = 2,

$$y_u = b + (n-1)\, d \sin \theta$$

$$y_l = -y_u$$

This is most easily seen by determining geometrically just where the stick center must be positioned in order to just produce a kill (y_l is depicted simply for ease of construction). As illustrated in Figure 22 there is a kill when

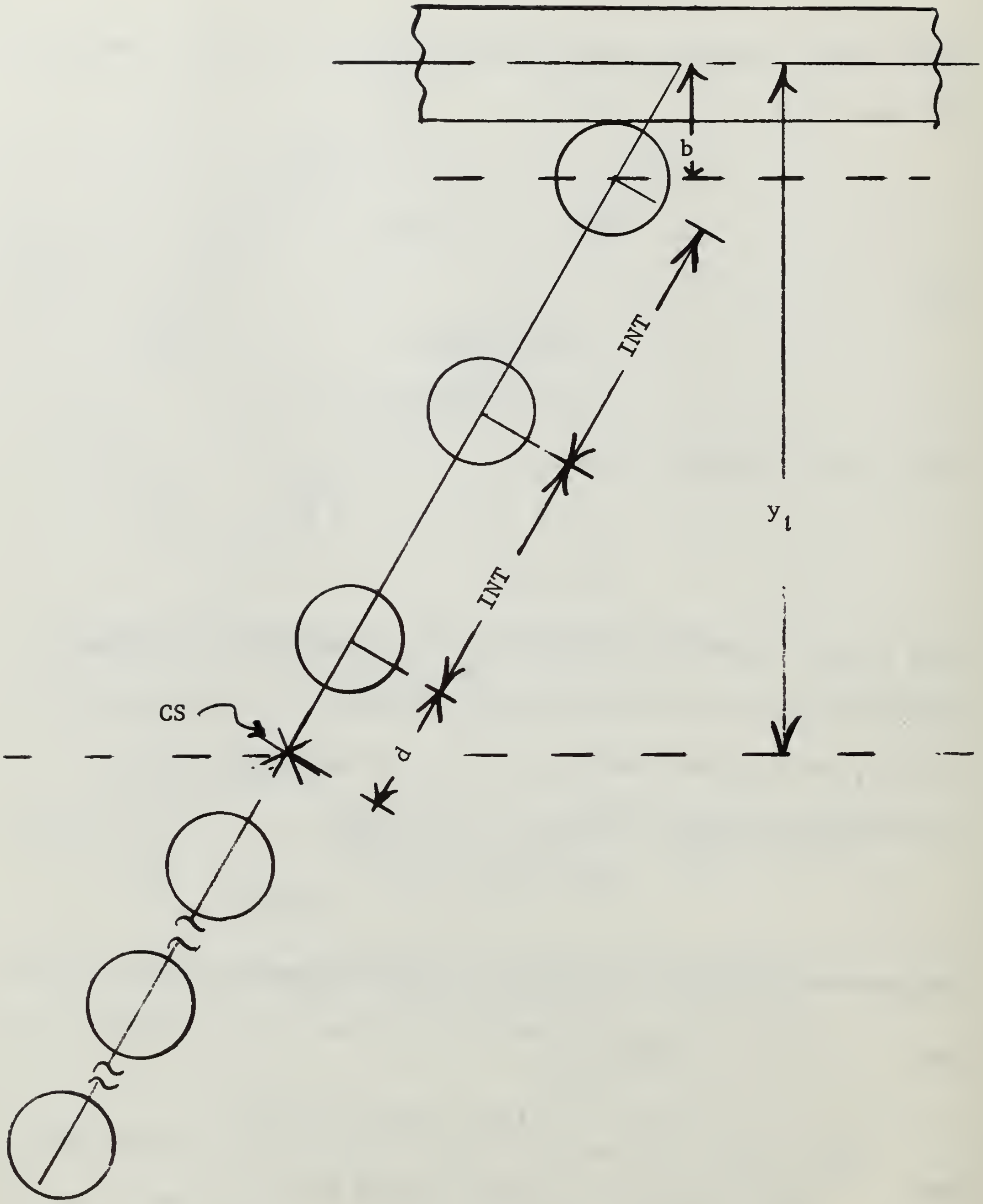
$$y_l \leq CS \leq y_u.$$

The distance involved is clearly $-\left\{\frac{(n-1)}{2}\left[(INT)/\sin \theta\right] + b\right\}$,

but $INT = 2d$

$$\therefore\ y_l = -\left\{\left[(n-1)\, d/\sin \theta\right] + b\right\}$$

and $y_u = \left\{(n-1)\, d/\sin \theta + b\right\}.$



Lower limit of CS producing a
target kill n = 6

FIGURE 22

APPENDIX C

LIMITS OF INTEGRATION BRIDGE TARGETS

For ease in notation assume $n = 2$, and $\theta \geq \beta$. The optimal bomb interval is then determined in exactly the same manner as for the railroad/road problem. That is

$$INT = (2\ EMD + BW)/\sin\theta = EBW/\sin\theta.$$

Next, the question of where the center of the stick can fall and still produce a kill is answered. Clearly until the EBL becomes a factor, y_u and y_ℓ are the same as for the railroad/road problem, namely

$$y_u = \frac{BW}{2} + EMD + d \sin\theta$$

$$= \frac{EBW}{2} + \frac{EBW}{2}\frac{\sin\theta}{\sin\theta} = EBW$$

$$y_\ell = -y_u = -EBW.$$

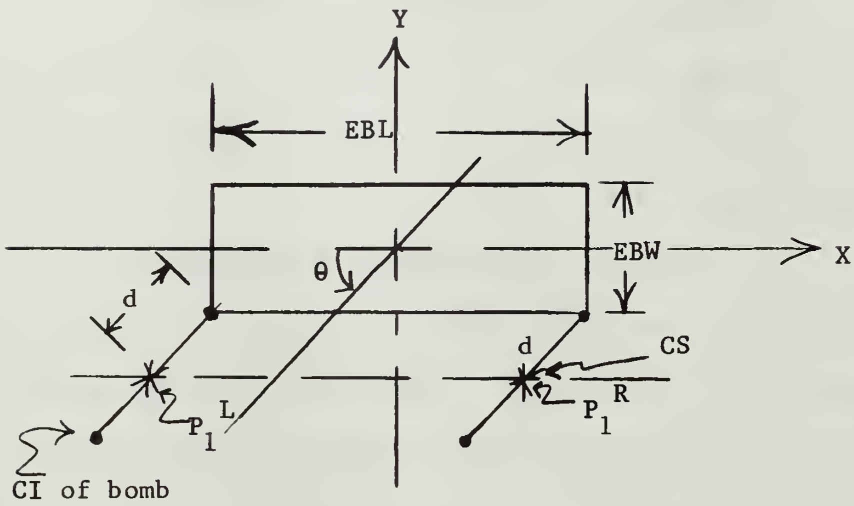
However, the effective bridge length affects the area of integration by defining right and left hand limit points in the X-direction. Consider y_ℓ, the left and right hand limits are determined by the most extreme position of CS when the forward most weapons just kills the target. See Figure 23. These two points, namely $P_1^R\ (X_1^R,\ Y_1^R)$ and $P_1^L\ (X_1^L,\ Y_1^L)$ where the superscript denotes left or right hand limit, can be represented by

$$X_1^L = -\left[\frac{EBL}{2} + d\cos\theta\right]$$

$$= -\left[\frac{EBL}{2} + \frac{EBW}{2}\frac{\cos\theta}{\sin\theta}\right] = -\left[\frac{EBL}{2} + \frac{EBW}{2}\operatorname{cotan}\theta\right]$$

$$X_1^R = \frac{EBL}{2} - d \cos \theta = \frac{EBL}{2} - \frac{EBW}{2} \text{ cotan } \theta$$

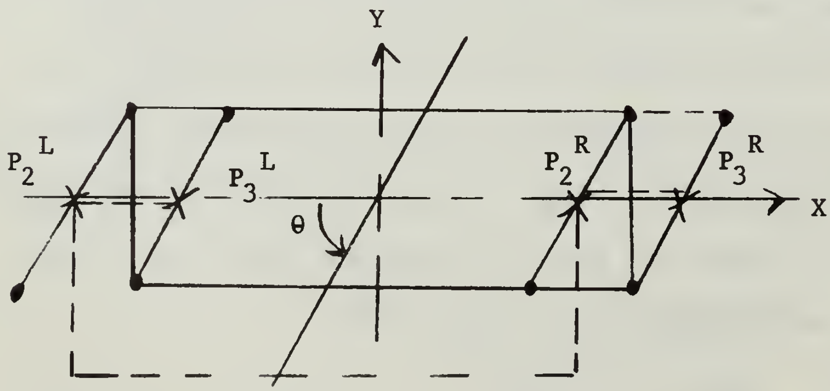
$$Y_1^R = Y_1^L = y_\ell .$$



Right and left limits of y_ℓ, n = 2

FIGURE 23

These two limits in the X-direction, X_1^R, X_1^L are then active until the CS coincides with the center line of the target. The limits of integration then change. See Figure 24.



Limits of integration, CS coinciding with center line of target

FIGURE 24

Now

$$X_2^L = -\left[\frac{EBL}{2} + d \cos \theta\right]$$

$$= -\left[\frac{EBL}{2} + \frac{EBW}{2} \text{ cotan } \theta\right]$$

$$Y_2^L = 0$$

$$X_3^L = - \frac{EBL}{2} + d \cos \theta$$

$$= - \frac{EBL}{2} + \frac{EBW}{2} \text{ cotan } \theta$$

$$Y_3^L = 0$$

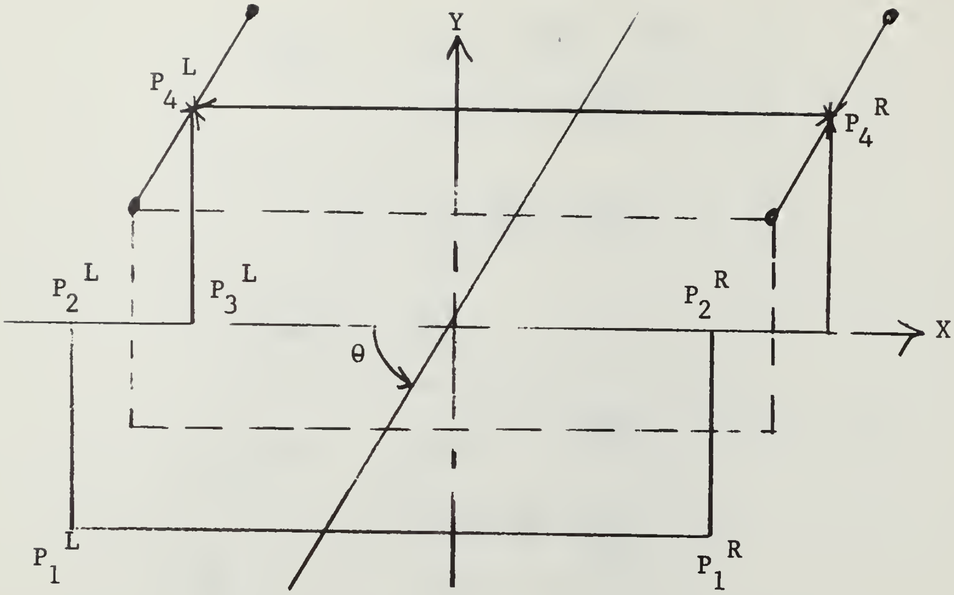
$$X_2^R = \frac{EBL}{2} - \frac{EBW}{2} \text{ cotan } \theta$$

$$Y_2^R = 0$$

$$X_3^R = \frac{EBL}{2} + \frac{EBW}{2} \text{ cotan } \theta$$

$$Y_3^R = 0.$$

The limits X_3^R, X_3^L hold until CS falls on y_u and this completes the polygon. See Figure 25.



Area of integration in X-Y coordinate system

FIGURE 25

P_4^L consists of $X_4^L = X_3^L$, $Y_4^L = y_u$ = EBW; and P_4^R consists of $X_4^R = X_3^R$, $Y_4^R = y_u$ = EBW.

The polygon is now well defined by the eight points (P_1^L, ..., P_4^L, P_1^R, ..., P_4^R) and by using the transformation

$$R = X \cos \theta + Y \sin \theta$$

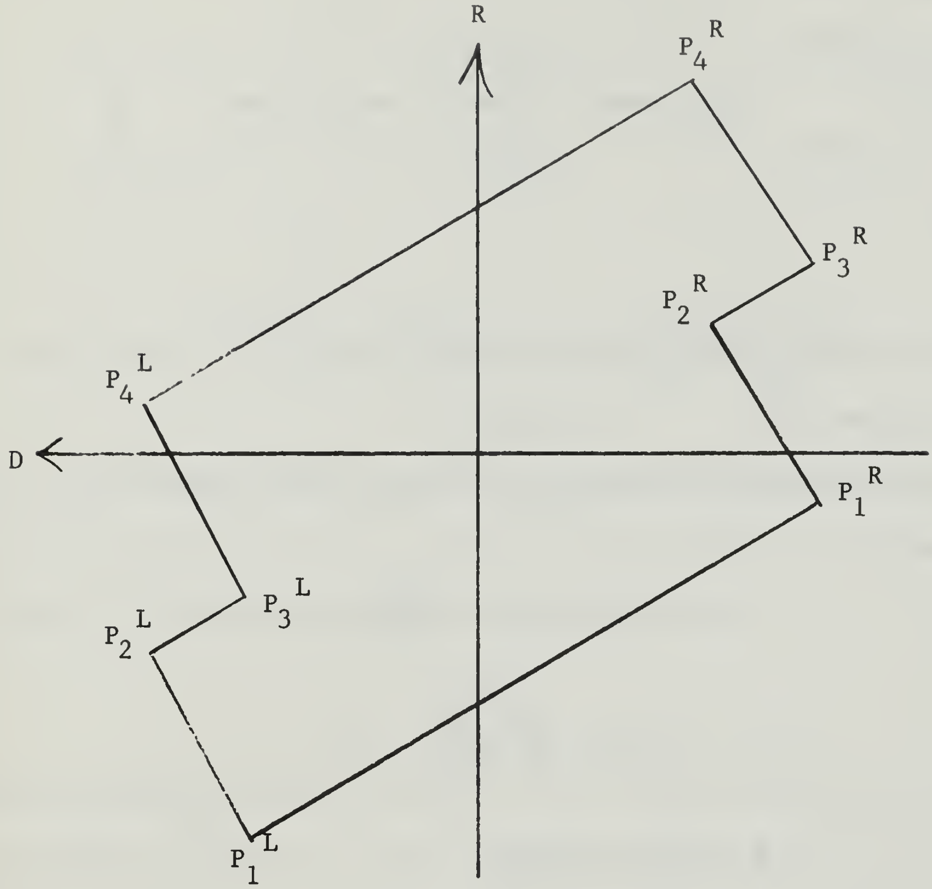
$$D = X \sin \theta + Y \cos \theta$$

the same polygon can be represented in the R-D coordinate system. See Figure 26.

The problem is now essentially the same as the railroad problem. That is to say

$$\text{Psph} = \iint\limits_{\substack{\text{area of}\\ \text{polygon}}} \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right]$$

and can be easily integrated by numerical techniques. See Appendix D. However, the limits of integration need to be systematically defined.



Area of integration in R-D coordinate system

FIGURE 26

The procedure used is to define the right and left hand limits of integration for the deflection error (D) by

$$D_L = m_j R + b_j \qquad j = 1, \ldots, 4$$

$$D_R = m_i R + b_i \qquad i = 1, \ldots, 4$$

where m_K and b_K are the appropriate slope and intercept for the portion of the boundary of the polygon involved in the integration.

The values of m_K and b_K can be calculated from the same formulae as in the railroad/road problem.

Therefore,

$$Psph = \int_{-\infty}^{\infty} \int_{D_R = m_i R + b_i}^{D_L = m_j R + b_j} \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right] dDdR$$

$$i = 1, \ldots, 4$$

$$j = 1, \ldots, 4;$$

and the appropriate m_K and b_K are used for the value of dR during the numerical integration.

One interesting note is that when $\theta = \beta$ the polygon collapses somewhat since

$$\cot\theta = \cot\beta = \frac{EBL}{EBW}$$

Hence

$$X_1^L = -\left[\frac{EBL}{2} + \frac{EBW}{2}\frac{EBL}{EBW}\right] = -EBL$$

$$X_1^R = \frac{EBL}{2} - \frac{EBW}{2}\frac{EBL}{EBW} = 0$$

$$X_2^L = -EBL$$

$$X_2^R = 0$$

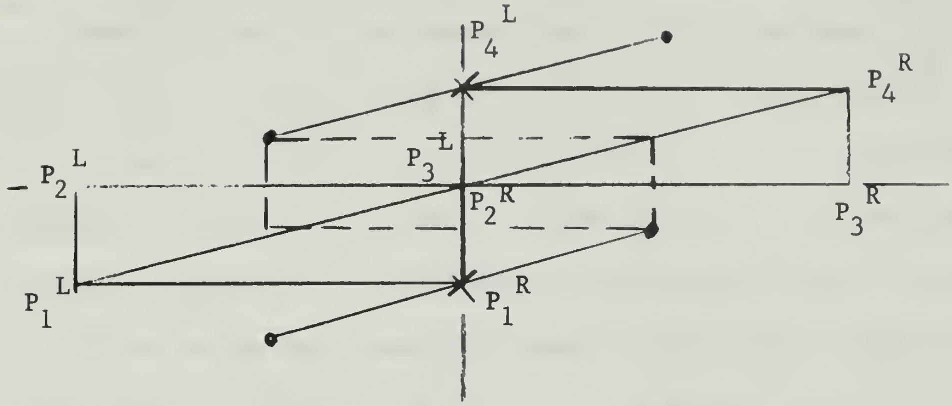
$$X_3^L = -\frac{EBL}{2} + \frac{EBW}{2}\frac{EBL}{EBW} = 0$$

$$X_3^R = \frac{EBL}{2} + \frac{EBW}{2}\frac{EBL}{EBW} = EBL$$

$$X_4^L = 0$$

$$X_4^R = EBL.$$

See Figure 27.



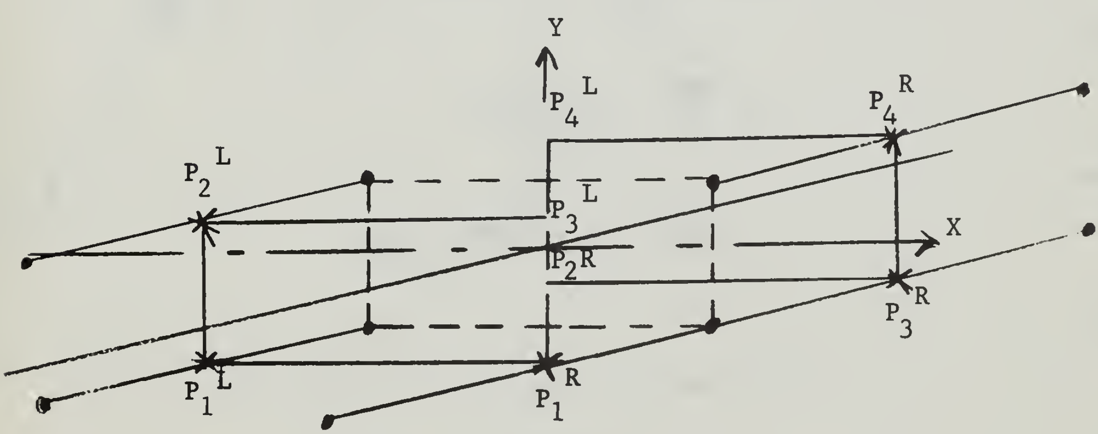
Area of integration X-Y coordinate system,
approach angle equals critical angle

FIGURE 27

Now suppose that the approach angle is less than β, this type of attack is exactly the same except the roles of EBL and EBW are reversed. The optimal bomb interval is now

$$INT = \frac{EBL}{\cos\,\theta}$$

and the formulae for the 8 points in the polygon are exactly the same except EBL is replaced by EBW and vice versa and due to the manner in which θ is measured cotan θ is replaced by tan θ, see Figure 28.



Area of integration, $\theta \leq \beta$

FIGURE 28

Once the polygon has been described the remaining development is exactly the same as when $\theta \geq \beta$ and, therefore, is not discussed. Notice, however, that when $\theta = \beta$ the formulae reduce to those obtained previously when $\theta = \beta$ (as they should). One implicit assumption in this development is that the target is assumed to be only of length BL. That is to say cratering the approaches to the bridge is considered a miss. This type of damage can easily be accounted for however by simply changing BL, or changing the lethality function.

The next problem is to extend this development to an arbitrary number of weapons. Suppose θ is greater than β, note that for each weapon in the stick there are 4 limit points, directly analogous to the four corners of the target. The polygon over which the integration is performed then must have 4n corners for any n. Note that the optimal bomb intervals are equal for a given θ. That is

$$\text{INT} = \text{EBW}/\sin\theta$$

and y_l and y_u are defined in the same manner as in the railroad/road problem

$$y_u = b + (n-1)\, d \sin\theta$$

where

$$b = \frac{\text{EBW}}{2},\quad d = \frac{\text{EBW}}{2\sin\theta}$$

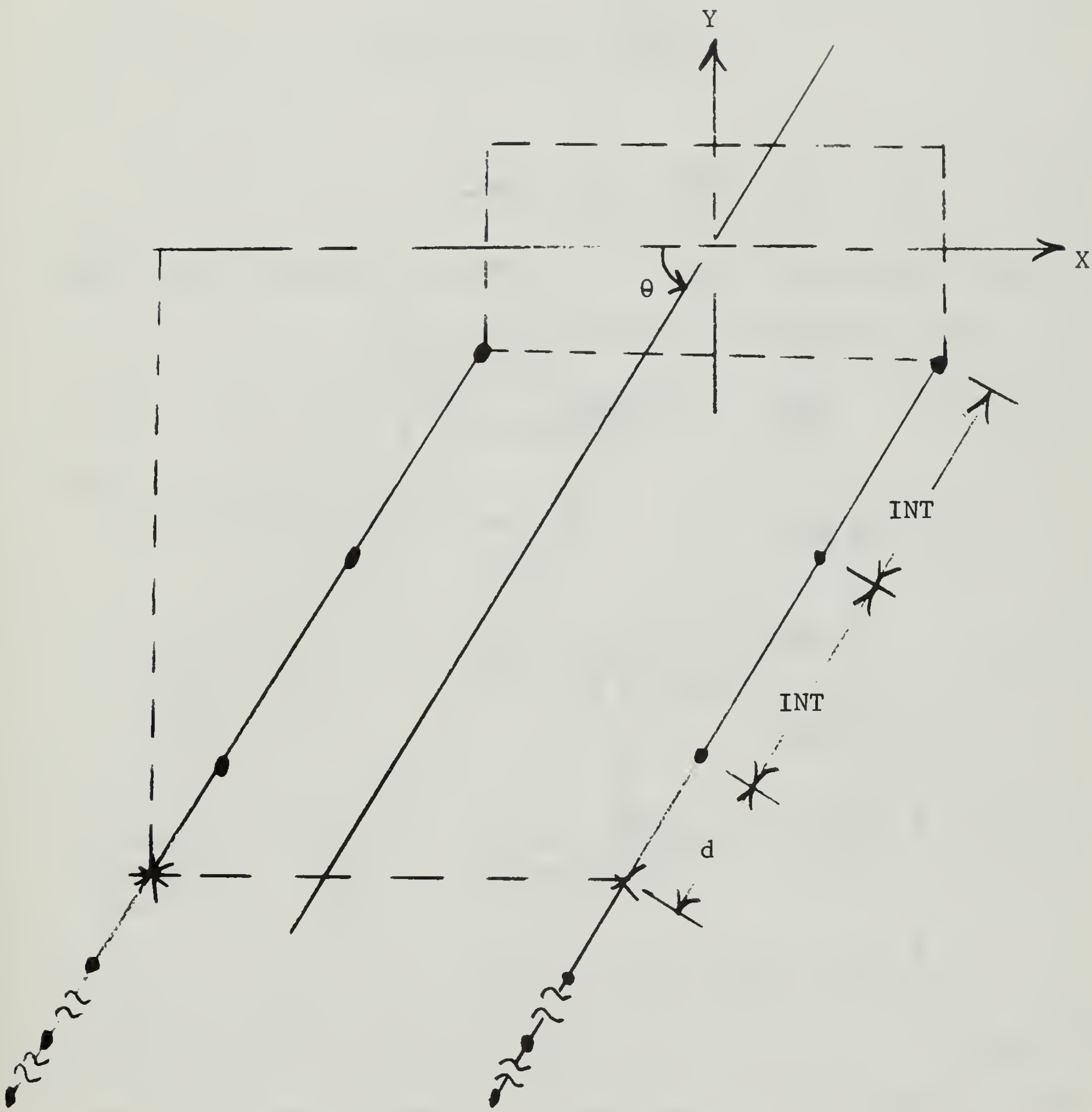
hence

$$y_u = \frac{\text{EBW}}{2} + (n-1)\,\frac{\text{EBW}}{2}\,\frac{\sin\theta}{\sin\theta}$$

and

$$y_l = -y_u.$$

Now the right and left hand limit points for y_l must be determined. They are defined by the most extreme position of CS when the most forward bomb just produces a kill. See Figure 29.



Limits of integration for y_l, n = 6

FIGURE 29

Clearly

$$X_1^L = -\left[\frac{EBL}{2} + \left(\left(\frac{n}{2} - 1\right) INT + d\right) \cos\theta\right]$$

$$= -\left[\frac{EBL}{2} + \left(\left(\frac{n-2}{2}\right) [2d] + d\right) \cos\theta\right]$$

$$= -\left[\frac{EBL}{2} + (n-1)\, d \cos\theta\right]$$

and

$$X_1^R = \frac{EBL}{2} - (n-1)\, d \cos\theta.$$

In a similar manner each point of the left and right hand limits can be constructed, using the following formulae:

$$X_{i+1}^L = X_i^L = -\left\{\frac{EBL}{2} + (n-s)\frac{EBW}{2} \operatorname{cotan}\theta\right\}, \quad s=1, 3, 5, \ldots, 2n-1$$
$$i=1, 3, 5, \ldots, 2n-1$$

$$Y_1^L = \frac{n}{2} EBW$$

$$Y_{i+1} = Y_i = \frac{P}{2} EBW, \quad P = -(n-2), -(n-4), \ldots, -2, 0, 2, \ldots n-2.$$
$$i = 2, 3, \ldots, 2n-1$$

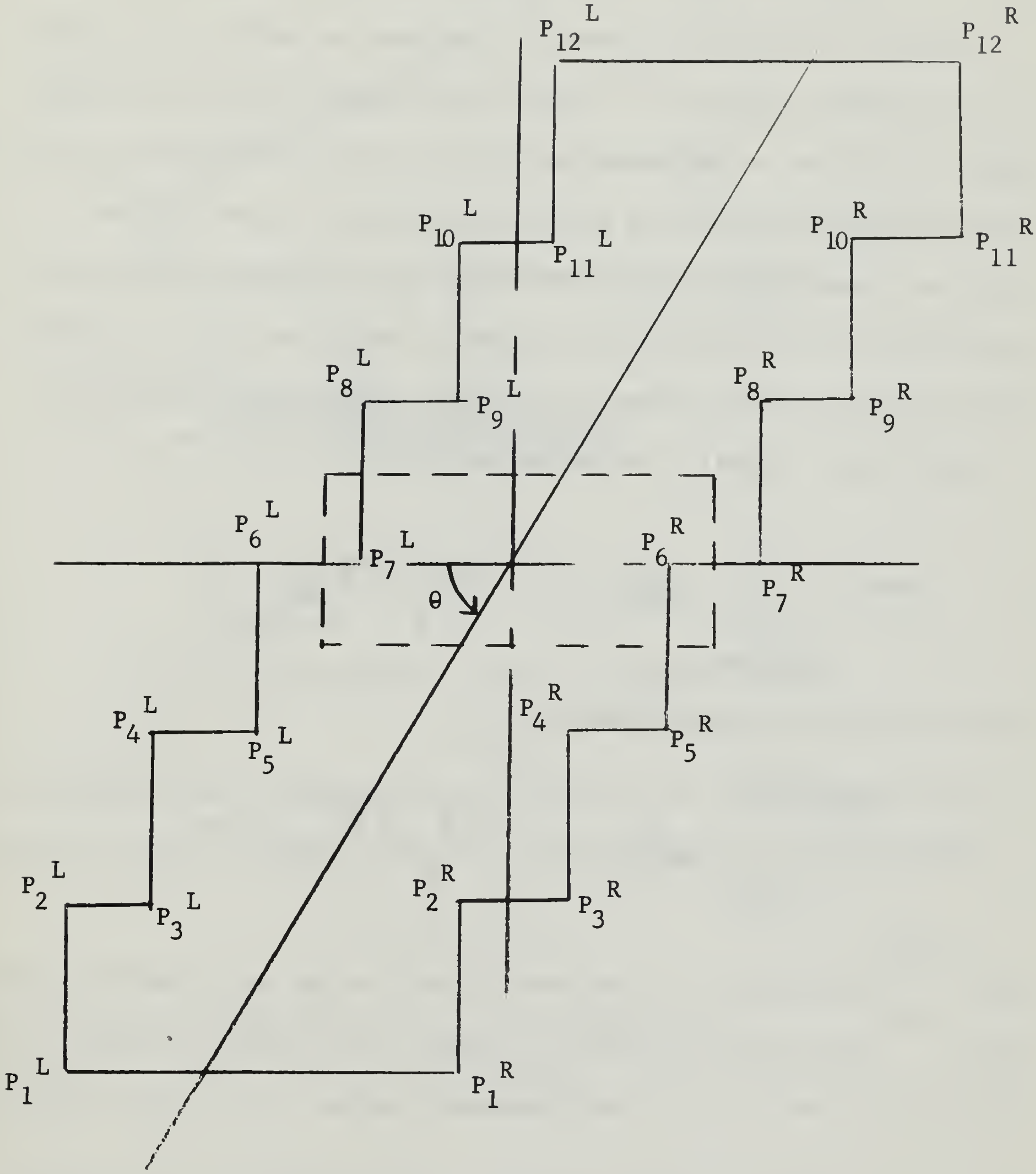
$$Y_{2n}^L = \frac{n}{2} EBW$$

$$X_{i+1}^R = X_i^R = \left[\frac{EBL}{2} - (n-s)\frac{EBW}{2} \operatorname{cotan}\theta\right], \quad s=1, 3, \ldots, 2n-1$$
$$i=1, 3, \ldots, 2n-1$$

$$Y_i^R = Y_i^L, \quad i=1, 2, 3, \ldots, 2n.$$

See Figure 30.

The formulae when $\theta < \beta$ once again are identical except for the replacement of EBW by EBL and vice versa and cotan θ by tan θ. Once the polygon for an arbitrary n has been defined in the X-Y coordinate system the remaining development is identical to the case for $\theta \geq \beta$. The only problem arising being the "bookkeeping" of the proper left and right hand limits in the numerical integration.



Area of integration for n = 6

FIGURE 30

APPENDIX D

NUMERICAL EVALUATION OF THE PROBABILITY INTEGRAL

In order to discuss the numerical integration method for a given set of parameter values (i.e., target dimensions, σ_R, σ_D, θ, n, EMD, $g_\theta(\theta)$) the optimal stick length and bomb interval is determined. The area of integration in the X-Y coordinate system is then defined using the formulae developed previously. This polygon is then rotated into the R-D coordinate system by the appropriate transformation and the following type of numerical integration is performed.

Recall the integral to be approximated

$$Psph = \iint\limits_{\substack{\text{area of}\\ \text{polygon}}} \frac{1}{2\pi\sigma_D\sigma_R} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right].$$

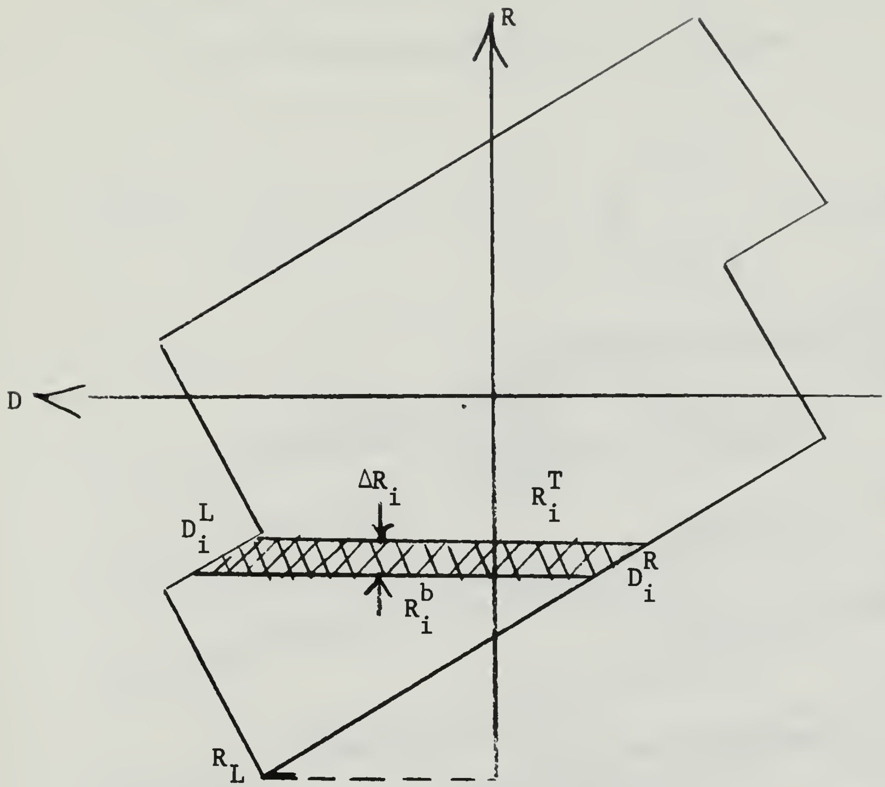
This integration is approximated by

$$Psph = \sum_{i=1}^{8\sigma_R/3} \int_{R_i^b}^{R_i^T} \int_{D_i^R}^{D_i^L} \frac{1}{2\pi\sigma_R\sigma_D} \exp\left[-\frac{1}{2}\left(\frac{R^2}{\sigma_R^2} + \frac{D^2}{\sigma_D^2}\right)\right] dDdR$$

where $\sum_{c=1}^{8\sigma_R/3}$ represents the addition of small strips of probability and $8\sigma_R/3$ determines the number of strips. For example, suppose the area of integration in the R-D coordinate system is as seen in Figure 31.

First, the lower limit R_L of integration is determined and $R^b{}_1$ is set equal to R_L

$$R^b{}_1 = R_L.$$



Area of integration in R-D coordinate system

FIGURE 31

The b denotes lower value of the ΔR_1 interval and 1 denotes the first interval in the summation. The upper limit of ΔR_1 is then defined by

$$R^T_1 = R^b + 3.$$

Next the center point R^*_1 of ΔR_1 is determined by

$$R^*_1 = \frac{R_1^T + R_1^b}{2}$$

and the right and left hand end points of D are determined:

$$D_1^R = m_i R^* + b_i$$

$$D_1^L = m_j R^* + b_j,$$

using the proper equation for the boundary of the polygon.

Next R^b_1 and R^T_1 are normalized and the

$$\text{PROB}\left[\frac{R_1^T}{\sigma_R} \leq R \leq \frac{R_1^b}{\sigma_R}\right] \text{ is computed by}$$

$$\text{PROB (1)} = \int_{R_1^b/\sigma_R}^{R_1^T/\sigma_R} \frac{1}{2\pi} e^{-\frac{u^2}{2}} du.$$

Next D_1^R and D_1^L are normalized and

$$\text{PROB}\left[\frac{D_1^R}{\sigma_D} \leq D \leq \frac{D_1^L}{\sigma_D}\right] \text{ is computed by}$$

$$\text{PROB (2)} = \int_{D_1^R/\sigma_D}^{D_1^L/\sigma_D} \frac{1}{2\pi} e^{-\frac{u^2}{2}} du.$$

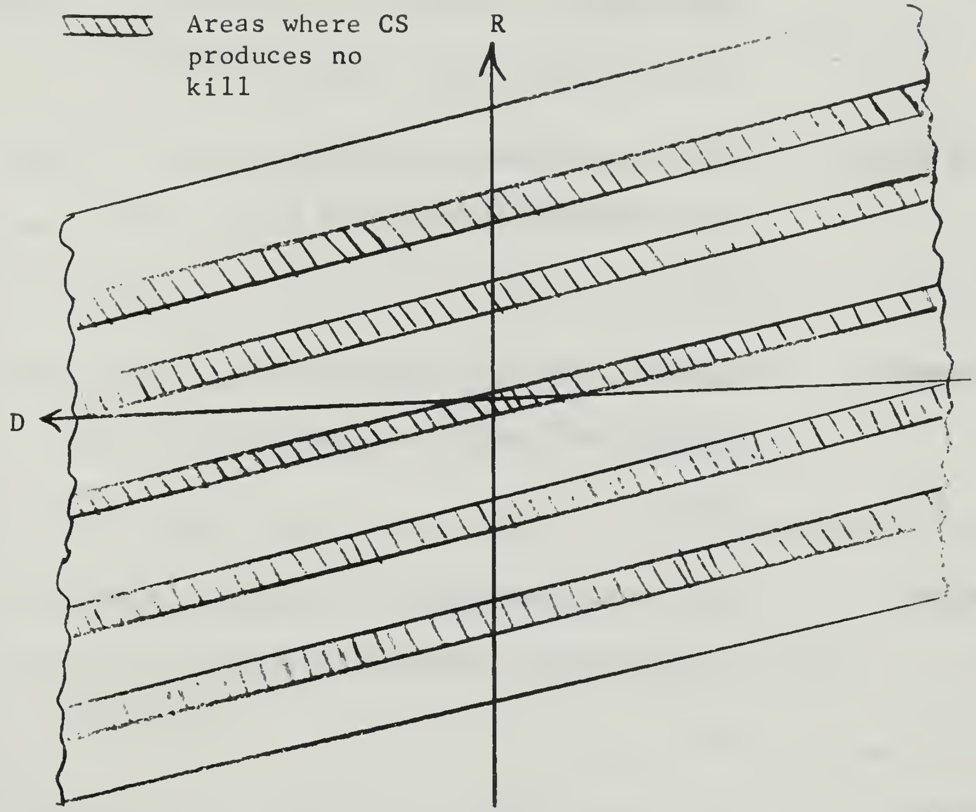
Next [PROB (2)] x [PROB (1)] is computed and saved.

The program then sets R_2^b equal to R_1^T and repeats the process adding the increase in probability to the previous total. This procedure is repeated until the uppermost limit of R is reached at which time the total value of the probability accumulated is printed. The entire procedure is repeated for each θ.

One further comment, in the railroad/road problem, since R theoretically varies from $-\infty$ to $+\infty$, the integration is limited to $\pm 4\sigma_R$.

The procedure is exactly the same when EAA is present but the area in the X-Y plane is slightly different. No problems are encountered when $\theta \leq \theta_0$. That is as long as the bomb interval used is less than or equal to the optimal bomb interval for the θ in the integration; however, when the bomb interval used exceeds the optimum bomb interval

(i.e. $\theta > \theta_0$) areas appear where the CS may fall and no kill occur. Figure 32 illustrates this condition for n = 6.



Area of integration EAA present, $\theta > \theta_0$

FIGURE 32

Formulae for the construction of this area are easily determined but since the computer program developed is a special case, that is the number of bombs is constant at 6, general formulae are not included. The problem was investigated only because questions arose about the validity of using errorless approach angles but since the results obtained were not greatly different than when θ was assumed perfectly known the investigation was discontinued.

APPENDIX E

GLOSSARY

AIR WING: Tactical unit aboard aircraft carrier made up of approximately 6 squadrons. Normally, the largest unit of aircraft that operate together.

APPROACH ANGLE: Acute angle measured from center line of target to horizontal projection of flight path during dive.

BOMB INTERVAL: Horizontal distance measured along flight path between the centers of impact of two adjacent bombs.

BOMB STICK: Line of bombs dropped at programmed intervals in a single dive. Normally falling in a straight line.

DIVE ANGLE: Acute angle measured in vertical plane between the horizontal and the flight path during the dive.

EFFECTIVE MISS-DISTANCE: Maximum distance any target dimension may be from the center of impact of the weapon and still produce a kill.

FLIGHT: Several airplanes operating together as a tactical unit.

GENERAL PURPOSE BOMBS:	Conventional ordnance constructed of high explosives and iron alloys. Produced in different weights and used on ordinary targets.
KILL THE TARGET:	With respect to a railroad/road target, kill implies cratering the road so as to make it temporarily impassable; for a bridge, collapsing the bridge is considered a kill.
LIGHT JET ATTACK AIRCRAFT:	Turbo-jet aircraft designed primarily to carry air to ground weapons. Much smaller in size than strategic bombers such as B-52, B-47.
PILOT MISS-DISTANCE DISTRIBUTION:	Probability distribution describing pilot induced errors between the intended point of impact of a weapon and its actual point of impact.
ROLL-IN:	Initiation of the tracking dive in a bomb run.
SINGLE PASS HIT:	Killing the target in one pass.
SQUADRON:	Tactical unit aboard aircraft carrier made up of approximately 15 aircraft of the same type. Smallest organizational unit.
STICK BOMBING:	Tactic involved in dropping several bombs during a single dive using programmed intervals.
STICK LENGTH:	Horizontal distance, measured along horizontal projection of flight path, from center of impact

of the first weapon released to center of impact of last weapon released during a single run.

TRACKING DIVE: Dive in which bombs are intended to be dropped. Pilot attempts to correct for all known errors and to place the geometric center of the bomb stick on the geometric center of the target.

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13. ABSTRACT

A mathematical model for the delivery of conventional bombs by stick bombing has been developed. This model considers bomb type, number of bombs, pilot miss-distance distribution, and dimensions of a rectilinear target. It may be used to estimate the effects of bomb interval and approach angle upon the single pass destruction probability of a target.

The significant variables of the stick bombing operation are identified, and functional relationships are developed among these. The model is used to examine the problem of determining an optimal approach angle and bomb interval for stick bombing bridges and railroads.

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Rectilinear targets						
Pilot miss-distance distribution						
Optimal approach angle						
Bomb interval						

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